ME 104: ENGINEERING MECHANICS II
Department of Mechanical Engineering
University of California at Berkeley

Fall Semester 2005
Thursday October 6 2005
Professor Oliver M. O'Reilly

First Midterm Examination
Closed Books and Closed Notes
Both Questions Carry Equal Points

Question 1
Loop-the-Loop

Early designs of roller coasters featured "loop-the-loop" with circular sections of track mated to straight sections (see Figure 1). In this problem, we wish to examine certain features of this design. To do this, we model the roller coaster cart as a particle of mass m moving on a track. We also assume that a dynamic Coulomb friction force, a drag force $-mk\|\mathbf{v}\|^3\mathbf{e}_t$, and a gravitational force $-mg\mathbf{E}_y$ act on the particle.

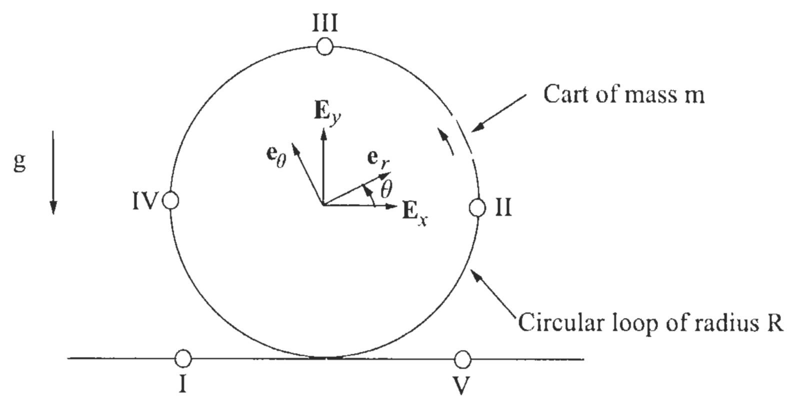


Figure 1: *Schematic of a roller coaster and a cart of mass m moving on it.*

(a) Depending on which section of track the particle lies on, its position vector is either

$$\mathbf{r} = x\mathbf{E}_x - R\mathbf{E}_y, \qquad \text{or} \qquad \mathbf{r} = R\mathbf{e}_r. \tag{1}$$

For both cases, derive expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. Clearly state any intermediate results that you use.

(b) For the five locations (labeled I, II, III, IV and V) on the roller coaster, draw the vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ and calculate $\mathbf{e}_b$.

(c) Draw a freebody diagram of the particle. Give clear expressions for the forces acting on the particle.

(d) Show that the equation governing the particle on the circular section with $\dot{\theta} > 0$ is

$$\ddot{\theta} = -\frac{g}{R}\cos(\theta) - kR^2\left(\dot{\theta}\right)^3 - \mu_k\left|\dot{\theta}^2 - \frac{g}{R}\sin(\theta)\right|. \tag{2}$$

(e) At the first instance where the circular and straight sections of track meet, how are $\dot{x}$ and $\ddot{x}$ related to $\dot{\theta}$ and $\ddot{\theta}$? Show that the change in resultant force experienced by the particle as it passes this point is proportional to $\frac{1}{R}$.

Question 2
A Friction Problem

A crate of mass m rests on a plane that is inclined at an angle α. A mechanism is in place that allows α to be varied as a function of time t: $\alpha = \alpha(t)$. The contact between the crate and surface is rough, with coefficients of static friction μ_s and dynamic friction μ_k. A spring of stiffness K and unstretched length L is attached to the crate.

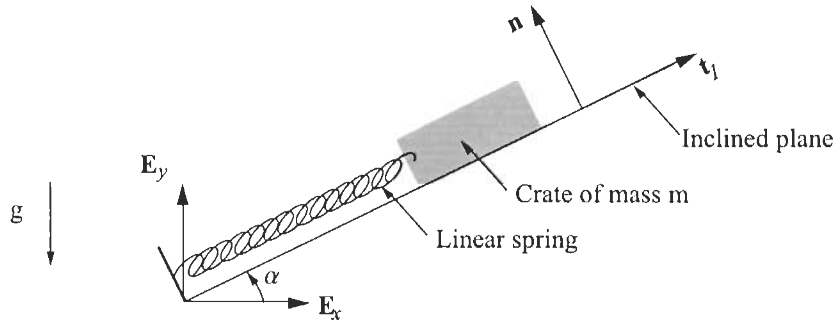


Figure 2: *Schematic of a crate of mass m moving on a rough inclined plane. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the crate.*

(a) Modeling the crate as a particle of mass m and negligible height, establish an expression for the position vector $\mathbf{r}$. Differentiating this vector, show that

$$\mathbf{a} = \left(\ddot{u} - u\dot{\alpha}^2\right)\mathbf{t}_1 + \left(u\ddot{\alpha} + 2\dot{u}\dot{\alpha}\right)\mathbf{n}. \tag{3}$$

The unit vectors $\mathbf{t}_1$ and $\mathbf{n}$ are shown in Figure 2.

(b) Draw freebody diagrams of the particle for the cases (i) when it is stationary relative to the inclined plane, and (ii) when it is in motion relative to the inclined plane. In your freebody diagram, give clear expressions for the friction and spring forces acting on the particle. Don't forget to specify the velocity vector of the particle relative to the inclined plane (i.e., $\mathbf{v}_{rel}$) if it is needed.

(c) When the particle is stuck to the plane, using $\mathbf{F} = m\mathbf{a}$, determine the friction force acting on the particle and the normal force $\mathbf{N}$ acting on the particle. Then, show that the static friction criterion is

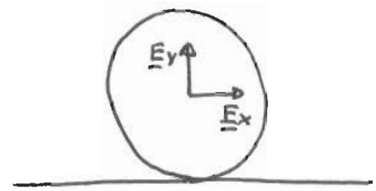
$$\left|K(u-L) - mu\dot{\alpha}^2 + mg\sin(\alpha)\right| \leq \mu_s \left|mu\ddot{\alpha} + mg\cos(\alpha)\right|. \tag{4}$$

(d) When the particle is in motion on the surface, show that the differential equation governing its motion is

$$m\left(\ddot{u} - u\dot{\alpha}^2\right) = -K(u-L) - mg\sin(\alpha) - \mu_k\|\mathbf{N}\|\frac{\dot{u}}{|\dot{u}|}. \tag{5}$$

In your solution, provide an expression for $\|\mathbf{N}\|$.

Problem I



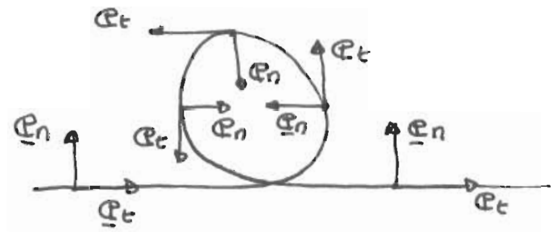
(a) on straight sections $\underline{r} = x\underline{E}_x - R\underline{E}_y$

$\Rightarrow \underline{v} = \dot{x}\underline{E}_x \; , \quad \underline{a} = \ddot{x}\underline{E}_x$

on circular section : $\underline{r} = R\underline{e}_r \Rightarrow \underline{v} = R\dot{\theta}\underline{e}_\theta \Rightarrow \underline{a} = R\ddot{\theta}\underline{e}_\theta - R\dot{\theta}^2\underline{e}_r$

where we have used the results $\dot{\underline{e}}_r = \dot{\theta}\underline{e}_\theta \; , \; \dot{\underline{e}}_\theta = -\dot{\theta}\underline{e}_r$.

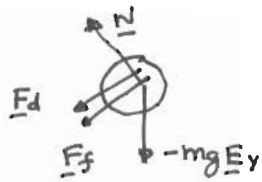
(b)



$\underline{e}_b = \underline{E}_z$
$= \underline{e}_t \times \underline{e}_n$
for all 5 points

[It suffices to calculate $\underline{e}_b$ @ one point here]

(c)



$\underline{F}_d = -mK\|\underline{v}\|^3\underline{e}_t$
$\underline{N} = N_n\underline{e}_n + N_b\underline{e}_b$
$\underline{F}_f = -\mu_K\|\underline{N}\|\underline{e}_t$

(d) From $\underline{F} = m\underline{a}$: on circular section : $\underline{e}_n = -\underline{e}_r \; , \; \underline{e}_t = \underline{e}_\theta \; , \; \underline{e}_b = \underline{E}_z$

$\cdot\underline{e}_r : \quad -N_n - mg(\underline{E}_y\cdot\underline{e}_r) = -mR\dot{\theta}^2$

$\cdot\underline{e}_\theta : \quad -mK\|\underline{v}\|^3 - \mu_K\|\underline{N}\|\beta - mg(\underline{E}_y\cdot\underline{e}_\theta) = mR\ddot{\theta} \quad , \; \beta = 1 \text{ as } \dot{\theta} > 0$

$\cdot\underline{E}_z : \quad N_b = 0$

Hence

$$\underline{N} = (mR\dot{\theta}^2 - mg\sin\theta)(-\underline{e}_r)$$

Thus

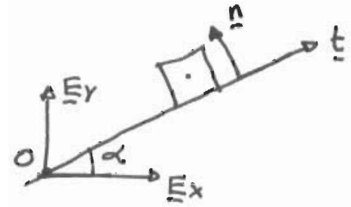
$$mR\ddot{\theta} = -mg\cos\theta - \mu_K|mR\dot{\theta}^2 - mg\sin\theta| - mK|R^3\dot{\theta}^3|$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{R}\cos\theta - \mu_K|\dot{\theta}^2 - g/R\sin\theta| - KR^2\dot{\theta}^3. \qquad (\dot{\theta} > 0)$$

(e) Going up : $\underline{v} = \dot{x}\underline{E}_x = R\dot{\theta}\underline{e}_\theta$, $\underline{a} = \ddot{x}\underline{E}_x \neq R\ddot{\theta}\underline{e}_\theta - R\dot{\theta}^2\underline{e}_r$
where $\underline{e}_\theta = \underline{E}_x$ and $\underline{e}_r = -\underline{E}_y$

Hence $\dot{x} = R\dot{\theta}$ and $\ddot{x} = R\ddot{\theta}$, but the acceleration vector is discontinuous
$\{\text{Change in Force}\} = \{\text{Change in Acceleration times mass}\} = -mR\dot{\theta}^2\underline{e}_r = m\frac{\dot{x}^2}{R}\underline{E}_y.$

Problem II



(a) $\underline{r} = u\underline{t}_1 + \underline{0}$

$$\Rightarrow \underline{v} = \dot{u}\underline{t}_1 + u\dot{\underline{t}}_1 = \dot{u}\underline{t}_1 + u\dot{\alpha}\underline{n}$$

$$\Rightarrow \underline{a} = \ddot{u}\underline{t}_1 + \dot{u}\dot{\underline{t}}_1 + \dot{u}\dot{\alpha}\underline{n} + u\ddot{\alpha}\underline{n} + u\dot{\alpha}\dot{\underline{n}}$$

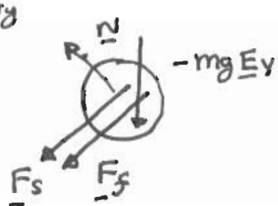
$$= (\ddot{u} - \dot{\alpha}^2 u)\underline{t}_1 + (u\ddot{\alpha} + 2\dot{u}\dot{\alpha})\underline{n}$$

$\underline{t}_1 = \cos\alpha\,\underline{E}_x + \sin\alpha\,\underline{E}_y$

$\dot{\underline{t}}_1 = \dot{\alpha}\underline{n}\,, \quad \dot{\underline{n}} = -\dot{\alpha}\underline{t}_1$

$\underline{n} = -\sin\alpha\,\underline{E}_x + \cos\alpha\,\underline{E}_y$

(b) Stationary



$$\underline{F}_s = -K(u - L)\underline{t}_1$$

$$\underline{F}_f = F_{f1}\underline{t}_1 + F_{fz}\underline{E}_z$$

$$\underline{N} = N\underline{n}$$

The freebody diagram when the crate is moving is identical, except that now

$$\underline{F}_f = -\mu_K \|\underline{N}\| \frac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|} \qquad \text{where } \underline{v}_{rel} = \dot{u}\underline{t}_1$$

(c) $\underline{F} = m\underline{a}$ where friction force is static and $\dot{u} = 0$ and $\ddot{u} = 0$.

$\cdot\underline{t}_1$: $\quad F_{f1} - K(u - L) - mg\,\underline{E}_Y\cdot\underline{t}_1 = -mu\dot{\alpha}^2$

$\cdot\underline{n}$: $\quad N - mg\underline{E}_Y\cdot\underline{n} = mu\ddot{\alpha}$

$\cdot\underline{E}_z$ $\quad F_{fz} = 0$

Hence :

$$\underline{F}_f = (K(u-L) + mg\sin\alpha - m\dot{\alpha}^2 u)\underline{t}_1$$

$$\underline{N} = (mg\cos\alpha + mu\ddot{\alpha})\underline{n}$$

The static friction criterion states that $\|\underline{F}_f\| \leq \mu_s\|\underline{N}\|$. Hence

$$|K(u-L) + mg\sin\alpha - m\dot{\alpha}^2 u| \leq \mu_s|mg\cos\alpha + mu\ddot{\alpha}|$$

(d) $\underline{F} = m\underline{a}$. Now $\quad \underline{F}_f\cdot\underline{t}_1 = -\mu_K\|\underline{N}\|\dfrac{\dot{u}}{|\dot{u}|}$

$\cdot\underline{t}_1$: $\quad -\mu_K\|\underline{N}\|\dfrac{\dot{u}}{|\dot{u}|} - K(u-L) - mg\sin\alpha = m(\ddot{u} - u\dot{\alpha}^2)$

$\cdot\underline{n}$ $\quad N - mg\cos\alpha = m(u\ddot{\alpha} + 2\dot{u}\dot{\alpha})$

Hence $\quad \|\underline{N}\| = |mg\cos\alpha + m(u\ddot{\alpha} + 2\dot{u}\dot{\alpha})|$