First Midterm Examination
Closed Books and Closed Notes

Question 1
A Particle on a Curve (20 POINTS)

As shown in Figure 1, a particle of mass m moves on a circular path of radius R.

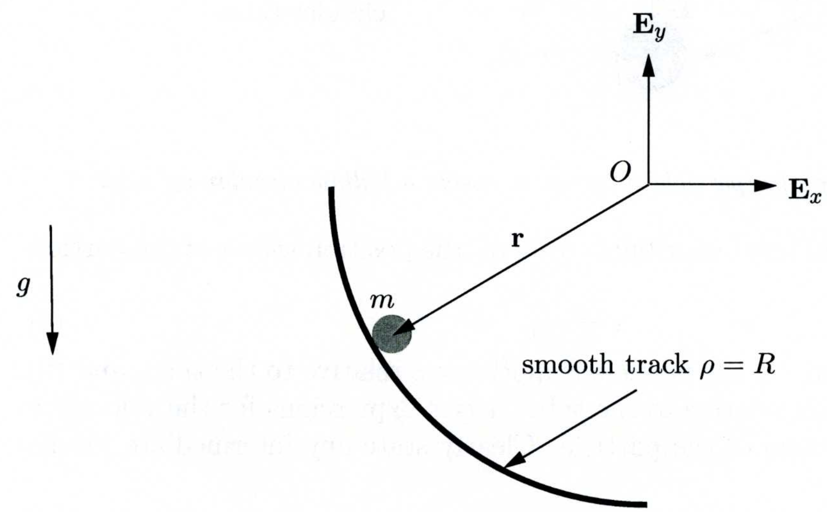


Figure 1: *Schematic of a particle of mass m moving on a smooth section of circular track. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle.*

(a) Starting from the usual representation for the position vector $\mathbf{r} = R\mathbf{e}_r$, show that the acceleration vector for the particle is

$$\mathbf{a} = R\ddot{\theta}\mathbf{e}_\theta - R\dot{\theta}^2\mathbf{e}_r. \tag{1}$$

(b) What are the unit tangent $\mathbf{e}_t$ and unit normal $\mathbf{e}_n$ vectors to the path of the particle? Illustrate your answers with a sketch.

(c) Draw a freebody diagram of the particle.

(d) Show that the normal force $\mathbf{N}$ acting on the particle and the acceleration of the particle are

$$\mathbf{N} = -mR\dot{\theta}^2\mathbf{e}_r + mg\sin(\theta)\mathbf{e}_r, \qquad \ddot{\theta} = -\frac{g}{R}\cos(\theta). \tag{2}$$

(e) Suppose the particle is initially placed at rest on the track with $\theta = \theta_0$. Show for the ensuing motion that

$$\frac{1}{2}\dot{\theta}^2(\theta) = -\frac{g}{R}\left(\sin(\theta) - \sin(\theta_0)\right), \qquad \mathbf{N} = mg\left(3\sin(\theta) - 2\sin(\theta_0)\right)\mathbf{e}_r. \tag{3}$$

Question 2

A Particle in a Tube (30 Points)

As shown in Figure 2, a particle of mass m is placed inside a cylindrical tube. One end of the tube is fixed to a drive shaft which is spun with an angular speed $\Omega(t)$. The inside of the tube is rough with a coefficient of static friction μ_s and a coefficient of kinetic friction of μ_k. In addition to friction and normal forces, a vertical gravitational force $-mg\mathbf{E}_z$ acts on the particle.

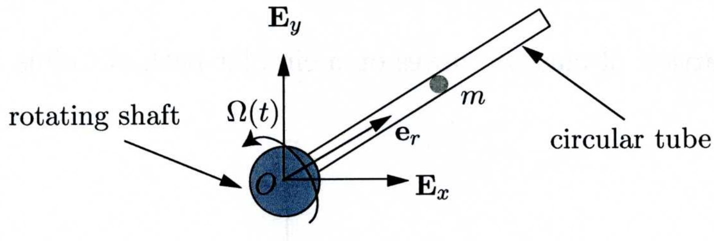


Figure 2: *Schematic of a particle of mass m inside a hollow circular cylinder.*

(a) Using a cylindrical polar coordinate system, the position vector of the particle is

$$\mathbf{r} = r\mathbf{e}_r. \tag{4}$$

For the two cases where (i) the particle is motionless relative to the tube, and (ii) the particle is in motion relative to the tube, derive expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. Clearly state any intermediate results that you use.

(b) Draw a freebody diagram of the particle. Give clear expressions for the forces acting on the particle, and distinguish the static friction and dynamic friction cases.

(c) Suppose that the particle is motionless relative to the tube. With the help of $\mathbf{F} = m\mathbf{a}$, show that the friction $\mathbf{F}_f$ and normal $\mathbf{N}$ forces acting on the particle are

$$\mathbf{F}_f = -mr\Omega^2\mathbf{e}_r, \qquad \mathbf{N} = mg\mathbf{E}_z + mr\dot{\Omega}\mathbf{e}_\theta. \tag{5}$$

(d) Suppose that the particle is in motion relative to the tube. Show that the differential equation governing r is

$$\ddot{r} - r\Omega^2 = -\mu_k\left(\sqrt{g^2 + \left(r\dot{\Omega} + 2\dot{r}\Omega\right)^2}\right)\frac{\dot{r}}{|\dot{r}|}. \tag{6}$$

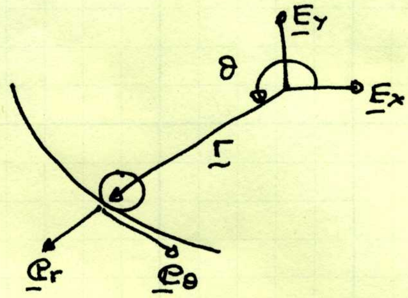
(e) Suppose that the particle is initially at rest relative to the tube: $r = R$. The tube in then spun up:

$$\Omega(t) = \alpha t, \tag{7}$$

where α is a constant. Show that after a time T the particle will start moving relative to the tube, where

$$T^2 = \frac{\mu_s}{\alpha^2}\sqrt{\frac{g^2}{R^2} + \alpha^2}. \tag{8}$$

Question 1



a) $\underline{r} = R\underline{e}_r$

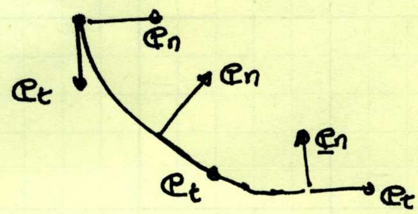
$\underline{v} = R\dot{\underline{e}}_r = R\dot{\theta}\underline{e}_\theta$

$\underline{a} = R\ddot{\theta}\underline{e}_\theta + R\dot{\theta}\dot{\underline{e}}_\theta$

$= R\ddot{\theta}\underline{e}_\theta - R\dot{\theta}^2\underline{e}_r$

$\dot{\underline{e}}_r = \dot{\theta}\underline{e}_\theta \quad , \quad \dot{\underline{e}}_\theta = -\dot{\theta}\underline{e}_r$

b)



$\underline{e}_n = -\underline{e}_r \qquad v = R\dot{\theta}$

$\underline{e}_t = -\underline{e}_\theta \qquad \dot{v} = R\ddot{\theta}$

c)

$\underline{E}_Y = \cos\theta\,\underline{e}_\theta + \sin\theta\,\underline{e}_r$

d) $\underline{F} = m\underline{a}$ $\qquad \cdot\underline{e}_r \quad -mR\dot{\theta}^2 = -mg\sin\theta + N$

$\cdot\underline{e}_\theta \quad mR\ddot{\theta} = -mg\cos\theta$

Hence $\underline{N} = (mg\sin\theta - mR\dot{\theta}^2)\underline{e}_r$

$\ddot{\theta} = -\frac{g}{R}\cos\theta$

e) Using $\int a(s)\,ds = \int v\,dv$ we find that $\dot{\theta}^2(\theta) - \dot{\theta}^2(\theta_0) = -\int_{\theta_0}^{\theta} \frac{2g}{R}\cos\eta\,d\eta$

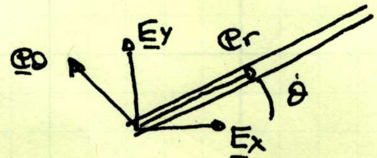
Hence $\frac{1}{2}\dot{\theta}^2(\theta) = -\frac{g}{R}\left(\sin\theta - \sin\theta_0\right)$

Substituting for $\ddot{\theta}^2$ in the expression for $\underline{N}$

$\underline{N} = mg\left(\sin\theta - (-2\sin\theta + 2\sin\theta_0)\right)\underline{e}_r$

$= \left(3mg\sin\theta - 2mg\sin\theta_0\right)\underline{e}_r$

QUESTION 2



(a) $\underline{r} = r\underline{e}_r$

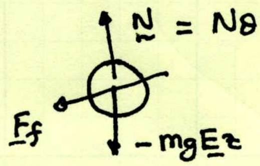
$$\underline{v} = r\dot\theta\underline{e}_\theta + \dot r\underline{e}_r = r\Omega\underline{e}_\theta + \dot r\underline{e}_r$$

$$\underline{a} = \dot r\dot\theta\underline{e}_\theta + r\ddot\theta\underline{e}_\theta + r\dot\theta(-\dot\theta\underline{e}_r) + \ddot r\underline{e}_r + \dot r\dot\theta\underline{e}_\theta$$

$$\theta = \int_{t_0}^{t}\Omega(\tau)\,d\tau + \theta(t_0)$$

$$= (\ddot r - r\dot\theta^2)\underline{e}_r + (r\ddot\theta + 2\dot r\dot\theta)\underline{e}_\theta = (\ddot r - r\Omega^2)\underline{e}_r + (r\dot\Omega + 2\dot r\Omega)\underline{e}_\theta$$

(b)



$$\underline{N} = N_\theta\underline{e}_\theta + N_z\underline{E}_z$$

Static: $\underline{F}_f = F_f\underline{e}_r\,,\quad |F_f| \le \mu_s\|\underline{N}\|$

Dynamic: $\underline{F}_f = -\mu_K\|\underline{N}\|\dfrac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|}$

$$\underline{v}_{rel} = \dot r\underline{e}_r = \underline{v} - r\Omega\underline{e}_\theta$$

(c) Static friction case $\quad \underline{v}_{rel} = \underline{0},\quad \underline{a} = -r\Omega^2\underline{e}_r + r\dot\Omega\underline{e}_\theta$

$\underline{F} = m\underline{a}$

$\cdot\underline{e}_r \qquad F_f = -mr\Omega^2$

$\cdot\underline{e}_\theta \qquad N_\theta = +mr\dot\Omega$

$\cdot\underline{E}_z \qquad N_z = mg$

Hence

$$\underline{F}_f = -mr\Omega^2\underline{e}_r \quad ; \quad \underline{N} = mg\underline{E}_z + mr\dot\Omega\underline{e}_\theta$$

(d) Dynamic Case :

$\cdot\underline{e}_r \qquad -\mu_K\|\underline{N}\|\dfrac{\dot r}{|\dot r|} = m(\ddot r - r\Omega^2)$

$\cdot\underline{e}_\theta \qquad N_\theta = m(r\dot\Omega + 2\dot r\dot\Omega)$

$\cdot\underline{E}_z \qquad N_z = mg$

Hence

$$\ddot r - r\Omega^2 = -\frac{\mu_K\|\underline{N}\|}{m}\frac{\dot r}{|\dot r|} \qquad \|\underline{N}\| = \sqrt{m^2g^2 + m^2(r\dot\Omega + 2\dot r\dot\Omega)^2}$$

(c)

$$\Omega(t) = \alpha t, \quad \Rightarrow \quad \dot{\Omega} = \alpha$$

Hence $$\underline{F}_f = -mR\alpha^2 t^2 \underline{e}_r$$

$$\underline{N} = mg\underline{E}_3 + mR\alpha \underline{e}_\theta$$

Static friction criterion $\|\underline{F}_f\| \leq \mu_s \|\underline{N}\|$

Particle will start moving when $\|\underline{F}_f\| = \mu_s \|\underline{N}\|$ at time $t = T$.

Hence
$$|-mR\alpha^2 T^2| = \mu_s \sqrt{m^2 g^2 + m^2 R^2 \alpha^2}$$

$$\Rightarrow \quad T^2 = \frac{\mu_s}{\alpha^2}\sqrt{\frac{g^2}{R^2} + \alpha^2}$$