Midterm 1, Section 2 and 3

1. [5 points] (a) When you bite into a pizza with extra cheese straight from the oven, you can burn your mouth. Biting into a piece of bread from the same oven is not so serious. Discuss the possible reasons for the difference.

total

+4 - the cheese has a greater thermal conductivity!! k, so the rate of heat transfer is greater w/ same T-difference

$H = \frac{Q}{\Delta t} = \frac{kA\Delta T}{l}$

+5 - cheese has greater k, and also higher C, heat capacity. More stored heat means more available to be transferred.

+2 - cheese has greater heat capacity

[5 points] (b) At the South Pole, when the Sun is well above the horizon and the wind speed is low, it is possible to take off your parka and work outside in shirt sleeves with the ambient temperatures at -30°C. Why is it possible? If the wind picks up even to a low level, it is hard to keep warm even in the parka. What does this example illustrate?

summed for each pt. included

+1 - possible b/c absorbing incident sun radiation

+2 - body heats up air, which is a poor conductor or good insulator → you stay warm

+1 - when wind picks up insulating warm layer is blown away and replaced by cold air → you get cold

+1 - This illustrates heat transfer by convection (and radiation) also wind-chill.

[5 points] (c) Compare the amount of internal energy between two blocks of copper, A and B. Block A is sitting on the table at a temperature T and Block B, at the same temperature, is on a moving conveyor.

External velocity has NO effect on internal energy.

sum

+4 - internal energies are equal b/c at the same temp

+1 - $E_{int,A} = E_{int,B}$ and realizing that no statement is made regarding the number of molecules. Thus you either needed to explicitly state you were assuming $N_A = N_B$ or show $U_A = \frac{3}{2} N_A k T_A$ $U_B = \frac{3}{2} N_B k T_B$ and that $\frac{U_A}{} = \frac{N_A}{}$

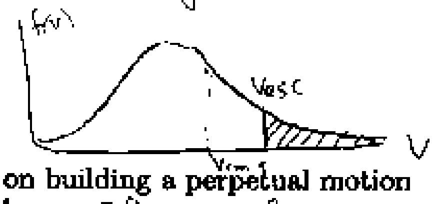
[5 points] (d) Explain why there is no hydrogen gas in the Earth's atmosphere even if it was present at the birth of this planet billion years ago.

Sum

+2 - H_2 gas is lighest and therefore $v_{rms} = \sqrt{\frac{3kT}{m}}$ is the greatest at the same temp T.

+2 - $v_{rms} \sim v_{esc}$ and H_2 must over time gain enough velocity to escape the gravitational pull of earth

+1 - depicted Maxwell-Boltz even if $v_{rms} < v_{esc}$, tail will have enough



[5 points] (e) What limits do the first and second laws place on building a perpetual motion machine and its performance? Graded holistically. If the first or second law were simply written w/no explanation, no points were given. If a grievous misstatement was made regarding 1° or 2° L.O.T. at least one point was taken off. Incoherent rambling, even if it possessed the criteria below, usually lost at least one point.

Sum w/ liberal amounts of interpretation for partial credit

Roughly:

+2 - The 1° law expresses conservation of energy which explicitly forbids getting any E for free. This makes a perpetual motion machine of the 1st kind impossible.
In other words you can't get something for nothing
$e \neq 100\%$

The 2° law expresses the change in entropy of the universe is always ≥ 0 or $\Delta S_{univ} \geq 0$, in reality none completely reversible

+1 - 2° law implies that you cannot ever have a machine that is 100% efficient at converting heat to work. Or you can never "break even". $e < 100\%$

+2 - This implies a PMM of the 2nd kind is impossible since it would require an infinite amout of energy to run forever.

Part (a) Solution

This problem is fairly straightforward. Either the temperatures must be converted to Celcius or the thermal conductivity must be changed accordingly. Most people converted the temperatures:

$$100^\circ\text{F} = 37.77\ldots{}^\circ\text{C} \approx 37.8^\circ\text{C}, \tag{1}$$

$$77^\circ\text{F} = 25^\circ\text{C} \tag{2}$$

The magnitude of the heat flow is given by the heat flow equation:

$$|H| = \frac{kA}{l}\,|\Delta T| = \left(10^{-3}\frac{\text{kW}}{\text{m-K}}\right)\frac{1\text{m}^2}{.006\text{m}}\,(37.8^\circ\text{C} - 25^\circ\text{C}) = 2.12\text{kW}. \tag{3}$$

The question also asks for you to explain the sign of the answer given. Many people used some version of the formula given on the test:

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}, \tag{4}$$

and managed to get a minus sign in their heat flow answer. Whatever formula or sign is used, the important point is that heat flows *into* the house from the outside. The meaning of the minus sign in equation (4) is that heat flows in the direction opposite the temperature gradient, or "downhill" from hot to cold.

Part (a) Grading Scheme

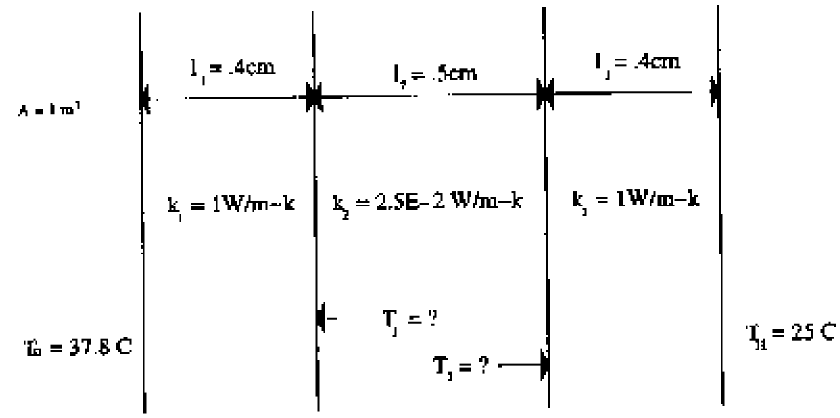
Points were awarded as follows, with some subjective judgement used as to the heinousness of errors.

- -3 No heat flow formula in some form, or no understanding of how to use it. [e.g. Did some sort of integration of equation (4)].
- -1 Messing up in some trivial way in evaluating the heat flow formula [e.g. Having a decimal point in the wrong place].
- -1 No explanation of sign, or saying that a negative sign means heat flows out of the house through the window and into the warmer outside air.
- -1.5 points were subtracted from the entire problem [(a) and (b)] if the temperatures were not converted to Celcius or Kelvin or converted incorrectly, but the rest of of the problem was consistent with this initial error. For part (a) assuming the the temperatures listed were in Celcius yielded $H = 3.83$kW

- For full credit, a heat flow equation in some form must be properly evaluated and whatever sign is obtained explained as heat flowing from hot to cold, or from the outside of the house to inside.

Part (b) Solution

The configuration for this problem is drawn below.



Both panes of glass are identical, with thermal conductivity k_1 and width l_1. I'm calling the thermal conductivity of the air k_2 and the distance between the panes l_2, the temperatures on the two inside faces of the window panes T_1 and T_2, the outside temperature T_o, and the house temperature T_H (see diagram).

Assuming we are in steady state, the rate of heat flow in each component between the outside and the inside of the house *must* be the same. If the situation were otherwise, parts of the system would be going up or down in temperature with time and we would not be in steady state. We have then that the the heat flow H is given by the following three expressions:

$$H = \frac{k_1 A}{l_1}(T_o - T_1), \tag{5}$$

$$H = \frac{k_2 A}{l_2}(T_1 - T_2), \tag{6}$$

$$H = \frac{k_1 A}{l_1}(T_2 - T_H). \tag{7}$$

Equations (5), (6), and (7), represent three equations for three unknowns, and can be solved straightforwardly. However, we only need to find the heat flow, and not necessarily the two inside temperatures. This expidites the algebra. Equations (5) and (7) say:

$$(T_2 - T_H) = (T_o - T_1)\,, \tag{8}$$

and equations (5) and (6) say

$$(T_1 - T_2) = \frac{k_1 l_2}{k_2 l_1}(T_o - T_1)\,. \tag{9}$$

Identically we have:

$$(T_o - T_1) + (T_1 - T_2) + (T_2 - T_H) = (T_o - T_H) \tag{10}$$

Which using (8) and (9) becomes:

$$2\,(T_o - T_1) + \frac{k_1 l_2}{k_2 l_1}(T_o - T_1) = (T_o - T_H) = 12.8°\text{C}. \tag{11}$$

$$\Rightarrow (T_o - T_1) = .246°\text{C}. \tag{12}$$

We can now put this result back into (5) to obtain the heat flow:

$$H = \frac{k_1 A}{l_1}(T_o - T_1) = \left(10^{-3}\frac{\text{kW}}{\text{m-K}}\right)\frac{1\text{m}^2}{.004\text{m}}.246°\text{C}. \approx 61.5\text{W} \tag{13}$$

There is another way to do this problem that is more efficient. It is also possible to realize that the heat flow equation is very similar to Ohms Law for electric current $V = IR$. With the substitutions $\Delta T \rightarrow V$ and $H \rightarrow I$, we can define the "thermal resistance":

$$R = \frac{l}{kA}, \tag{14}$$

and then:

$$HR = \Delta T. \tag{15}$$

Here we simply have resistors in series. The equivalent resistance of the series is given by.

$$R_{tot} = R_1 + R_2 + R_3 = 2\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} = .208\frac{\text{K}}{\text{W}} \tag{16}$$

The heat flow is then simply given by:

$$H = \frac{\Delta T}{R_{tot}} = \frac{37.77°\text{C} - 25°\text{C}}{.208\text{K/W}} \approx 61.5W \tag{17}$$

The final answers here and in (13) are *extremely* dependant on how many signifigant figures were kept when initially converting the temperatures, so a broad range of final numbers were accpepted as long as the method was reasonable.

Part (b) Grading Scheme

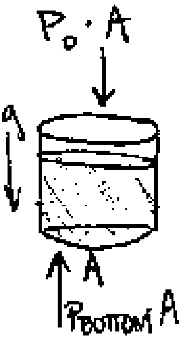
Points were awarded as follows, with some additional judgement used as to the seriousness of errors not listed here.

- +5 for recognizing that the heat flow was the same for each segment, and setting up equations equivalent to (5), (6), (7).
- +6 <u>or</u> +7 points for working from the heat flow equations and making some progress, but not getting to an answer [e.g. Finding one of the intermediate temperatures].
- +8 points for getting an answer working from the heat equations, but failing to get the right answer somehow, either do to an algebra mistake, a calculation error, or both.
- +2 points for realizing the heat flow should be the same, but failing to set up equations.
- +3 Used the steady state approach to find temperatures, but then added heat flows in the end.
- +3 points for using a formula equivalent to the resistance approach [see above] *without* any justification, but failing to get the right answer [e.g. remembered some formula and plugged in wrong].
- +6 points for using the resistance approach *without* any justification and getting the right answer. [e.g. remembered some formula and plugged in correctly].
- +7 points for the correct answer using the resistance approach *with some* attempt at justification, but pretty weak. [e.g. remembered some formula and plugged in correctly, and wrote "steady state" or something else equally apathetic].
- +8 points for a well justified resistance approach with a silly error.
- Only one point for simply adding the heat flows of the three components, pretending that they are effectively in parallel. This shows a strong lack of understanding of this problem, especially since the heat flow calculated is much more than in part (a), and then what would be the point of having a double-paned window.
- +2 points for adding heat flows (as above), realizing this was wrong, and stating so on the test.

- +3 points for realizing that the heat flow is constant, but only setting up one middle temperature and failing to get anywhere.
- +4 points for a *justified* approximation of putting the whole temperature difference across the air in between the panes, but then getting the wrong answer.
- +6 points for a *justified* approximation of putting the whole temperature difference across the air in between the panes, and then getting close to the right answer. This gets close to the correct answer, but misses the spirit of the problem, which can be solved exactly.
- -1.5 points were subtracted from the entire problem (a) and (b) if the temperatures were not converted to Celcius or Kelvin or converted incorrectly, but the rest of of the problem was consistent with this initial error. For part (b), assuming that the temperatures listed were given in Celcius yielded $T_1 = 99.56^0C$ and $H = .11$kW.
- +10 points for the correct answer with a good justification of either the resistance approach or the "same heat flow" approach.
- +10 points: Converted temperatures wrong in part (a), but then was totally consistent with with the incorrect temperatures.

SOLUTIONS

3. [15 points] A kettle of water is about to boil at 100°C. Steam bubbles are formed at the bottom of the kettle and some of them rise to the surface. The depth and the density of water are H and ρ respectively. The pressure at the surface of the water is P_0.



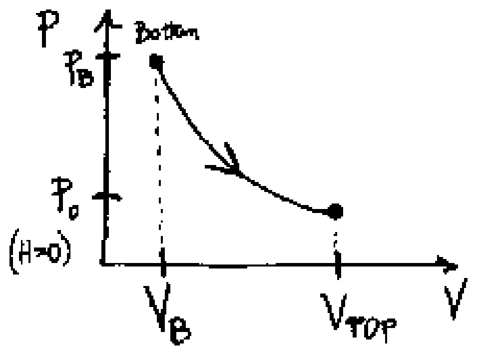
Scheme: Right answer: 1 point
One Mistake: 0.5 point

(a) [1 points] What is the pressure at the bottom of the kettle?

$$P_{BOTTOM} \cdot A = P_{AIR} \cdot A + m_{WATER}\, g$$

$$P_{BOTTOM} = P_{AIR} + \frac{\rho A H g}{A} = \boxed{P_{AIR} + \rho H g} = \boxed{P_0 + \rho H g}$$

(b) [4 points] By treating the steam inside the bubbles as an ideal gas, draw the PV diagram and specify the initial and final states as well as the path of the bubble in your plot if it is in thermal equilibrium with the water.



Thermal Equilibrium $\Rightarrow$ same temperature = 100° C

$\Rightarrow$ Isothermal expansion

So, $NKT = \text{const} = PV$

$$P_B V_B = P_0 V_{TOP}$$

In terms of V_B,

$$V_{TOP} = \frac{P_B V_B}{P_0} = \frac{(P_0 + \rho H g)}{P_0} V_B$$

Grading Scheme

- 2 pts. for graph (direction AND shape)
- 1 pt. for noticing isothermal nature (can appear in other parts)
- 1 pt. for correct value of V_{TOP} (can appear in other parts)

(b) [5 points] Starting from the definition of work done for a gas, calculate the total amount of work done by the steam in a bubble that rises from the bottom of the kettle to the surface.

Definition: $W_{BY} = \int P\,dV$

$$= \int_{V_B}^{V_{TOP}} \frac{NKT}{V}\, dV \quad \text{because } PV = NKT \Rightarrow P = \frac{NKT}{V}$$

$$= \int P_B V_B \frac{1}{V}\, dV$$

$$= P_B V_B \int_{V_B}^{V_{TOP}} \frac{dV}{V}$$

$$= P_B V_B \ln\left(\frac{V_{TOP}}{V_B}\right)$$

$$\boxed{W = P_B V_B \ln\left(1 + \frac{\rho H g}{P_0}\right) = NKT \ln\left(1 + \frac{\rho H g}{P_0}\right)}$$

Grading Scheme

- 1 pt. $W = \int P\,dV$
- 3 pts: Consistent calculation of W (dealing with integral, using $P = \frac{NKT}{V}$, etc.)
- 1 pt. Final Answer

(c) [3 points] What is the change in entropy of the steam bubble?

$\Delta S = \int \frac{dQ}{T} = \frac{1}{T}\int dQ = \frac{Q}{T}$

But $T = \text{const} \Rightarrow \Delta U = 0 \Rightarrow Q = W$

So, $\Delta S = \frac{W}{T} = \boxed{NK \ln\left(1 + \frac{\rho g h}{P_0}\right)}$

Grading Scheme
- 1 pt. for ΔS equation
- 1 pt. for calculation
- 1 pt. for answer.

NOTE: Simply writing $\Delta S = \frac{d}{2} NK \ln\left(\frac{T_f}{T_i}\right) + NK \ln\left(\frac{V_f}{V_i}\right)$ will not receive full credit

(d) [2 points] What is the change in entropy of the universe for this process? Explain your answer

$\boxed{\Delta S = 0}$ while the steam bubble rises to the surface. This is a reversible process because the steam bubble is always in thermal equilibrium (and you can plot it on a P-V diagram). Other equivalent justifications are that the Q into the bubble = the Q out of the steam.

Or, $\Delta S_{universe} = \int \frac{dQ_{steam}}{T} + \int \frac{dQ_{water}}{T}$

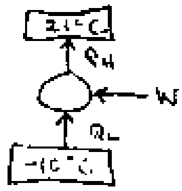
$= \frac{Q_{steam}}{T} + \frac{Q_{water}}{T}$

$= \frac{Q_{steam}}{T} - \frac{Q_{steam}}{T} = 0.$

Grading Scheme:
- 1 pt. for $\Delta S = 0$
- 1 pt. for explanation

Note: point values in ( )

Problem 4 – refrigerator, $\frac{dW}{dt} = \frac{3}{4}\text{hp}$.



(3 total) a) CP_{max} = ?

(1) $CP = \frac{Q_L}{W} = \frac{Q_L}{Q_H - Q_L}$ $\left(\frac{1}{2}\right)$ [Carnot cycle is max efficiency so $\frac{Q_L}{T_L} = \frac{Q_H}{T_H}$ or $\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$]

(1) $CP_{max} = \frac{T_L}{T_H - T_L}$

$\left(\frac{1}{2}\right)$ $CP_{max} = 8.48$

* Note $Q_L \neq T_L$?! * ← consider units

(4 total) b) operating at 43% of max CP, how much heat exhausted to environment per sec.

(1) $CP = .43\, CP_{max} = .43(8.48) = 3.65$ (relate CP and CP max)

(1) $CP = \frac{Q_L}{W}$, $3.65 = \frac{Q_L}{W}$ (relate Q_L, CP)

(1) $Q_L = Q_H - W$, so $3.65 = \frac{Q_H - W}{W} = \frac{Q_H}{W} - 1$ (relate Q_L, Q_H, W)

$\frac{Q_H}{W} = 4.65$

$\left(\frac{1}{2}\right)$ $\frac{dQ_H}{dt} = 4.65\, \frac{dW}{dt}$ (realizes we are talking about rates)

or

Q_H in one sec. = 4.65 W in one sec.

(so final answer either Q_H in one sec. in J or $\frac{dQ_H}{dt}$ in J/s)

$\left(\frac{1}{2}\right)$ $\frac{dQ_H}{dt} = 2601.7$ J/s (solve equation, get right answer)

* Notes:

→ no explicit understanding shown that we are talking about rates -½

→ uses a wrong CP_{max} from (a): no penalty

→ solves for Q_L. 2.5 pts. max

→ solves for Q_L but calls it Q_H 1.5 pts. max

(4 total) c) How long to cool and freeze water (4.2 kg) at 18°C to 0°C? (in the freezer)

(1) $Q_{H_2O\ water} = m_w c_w \Delta T$ } (includes right sign values + plugging in correctly)

(1) $Q_{water \to ice} = m_w L_f$

$Q = 4.2\text{ kg}\left(\frac{1\text{ kcal}}{\text{kg}\cdot{}^\circ\text{C}}\right)(0^\circ\text{C} - 18^\circ\text{C}) - 4.2\text{ kg}(80\text{ kcal/kg})$

$\left(\frac{1}{2}\right)$ $Q = -411.6$ kcal $= -1.72\times10^6$ J (math comes out right)

Q_L = heat out of freezer

(1) $CP\left(\frac{dW}{dt}\right) = \frac{dQ_L}{dt}$

$\frac{dQ_L}{dt} = .43(8.48)\frac{3}{4}hp\left(\frac{746\ J/s}{hp}\right) = 2040\ J/s$

$\left(\frac{1}{2}\right)$ $Q = \frac{dQ_L}{dt}(\text{time})$

$1.72\times10^6 J = 2040\frac{J}{s}(t)$, $t = 843\,s \cong 14$ min

* Notes:

→ if found + used Q_L from pt. (b) that's okay.

→ (−1) if use Q_H as rate out of freezer.

→ (−1.5) if use some expression like time = $\frac{Q}{\text{Work}}$

(4 total) d) What is ΔS_{ice}, ΔS_{room}, $\Delta S_{universe\ \underline{min}}$?

$\Delta S = \int \frac{dQ}{T}$

$\Delta S_{ice} = \Delta S_{18\to0} + \Delta S_{freeze}$

$\left(\frac{1}{2}\right)$ $\Delta S_{18\to0} = \int_{18+273}^{0+273} \frac{mc\,dT}{T}$

$\left(\frac{1}{2}\right)$ $\Delta S_{freeze} = \frac{-m_w L_f}{T}$

$\left(\frac{1}{2}\right)$ signs + plugging in: $\Delta S_{ice} = m_w c_w \ln\frac{T_f}{T_i} - \frac{m_w L_f}{273}$

$= 4.2kg\left(\frac{1kcal}{kg\cdot K}\right)\ln\left(\frac{273}{291}\right) - \frac{4.2kg\,(80\,kcal/kg)}{273°K}$

$= -0.267$ kcal/K $-$ 1.23 kcal/°K $= -1120\frac{J}{K} - 5144.6$

$\boxed{\Delta S_{ice} = -1.497\ kcal/°K = -6264.85\ J/°K}$

$\left(\frac{1}{2}\right)$ $\Delta S_{room} = \frac{Q_H}{T}$

$\left(\frac{1}{2}\right)$ $= \frac{\left(\frac{dQ_H}{dt}\right)(t)}{T}$

$\Delta S_{room} = \frac{(2601.7\ J/s)(843s)}{294\ K} = 7460\ J/K = 1.78\ kcal/K = \Delta S_{room}$

(1/2) math for whole problem.

$\left(\frac{1}{2}\right)$ $\Delta S_{universe\ min} = \Delta S_{ice} + \Delta S_{room} = \boxed{1195\ J/K = \Delta S_{universe(min)}}$

$\left(\frac{1}{2}\right)$ $\Delta S_{universe} \geq \Delta S_{ice} + \Delta S_{room}$ because the engine that runs the refrigerator also contributes entropy to the universe (real engines are irreversible)

*Note

→ the last 1/2 point was missed by everyone.

→ to get credit for answers that have the wrong sign (based on algebraic mistakes, etc.) you must point out why the sign is wrong.

5. [15 points] A gas mixture in a container of fixed volume V, made up of two different kinds of atoms, is in thermal contact with a heat bath at temperature T. There are N_1 atoms of the first kind, each has a mass of m_1. For the second type, there are N_2 atoms, each has a mass m_2.

(a) [4 points] Calculate the total pressure exerted by the gas mixture.

$P_1 = \frac{N_1 kT}{V}$, $P_2 = \frac{N_2 kT}{V}$ from ideal gas law. +1 depends only on N, T, V

$P_{tot} = \frac{F_{tot}}{A} = \frac{F_1 + F_2}{A} = P_1 + P_2 = \boxed{\frac{kT}{V}(N_1 + N_2)}$ +2 $P_{tot} = P_1 + P_2$

+1 final answer

no mass dependence!

(b) [3 points] What is the ratio of the pressure exerted by the first kind of atoms to that of the second kind?

trivially from a), $\frac{P_1}{P_2} = \frac{\frac{N_1 kT}{V}}{\frac{N_2 kT}{V}} = \boxed{\frac{N_1}{N_2}}$ +2 if correct given answer to a)

+1 final answer

(c) [5 points] Now suppose $N_1 = N_2 = N$, and a chemical reaction takes place that combines the two kinds of atoms to form diatomic molecules of mass m_1+m_2. After the temperature is back to T, what would be the ratio of the pressure after combination to that before the reaction?

Initially, $N_i = N_1 + N_2 = 2N \Rightarrow P_i = \frac{2NkT}{V}$ +1 correct P_i

after reaction, only N molecules $\Rightarrow P_f = \frac{NkT}{V}$ +2 $N_f = N$

+1 correct P_f

So $\boxed{\frac{P_i}{P_f} = 2.}$ +1 correct ratio

-1 if correct, given answer to a).

doesn't matter that final state diatomic

(d) [3 points] If the temperature of the heat bath is changed to T', what is the difference in the amount of heat going into (i) the original gas mixture, and (ii) the final gas system?

$\Delta E_{int} = Q - W$ (1st Law)

Volume fixed, so $W = 0$

$\Rightarrow Q = \Delta E_{int} = \frac{d}{2} Nk\Delta T$

– original gas mixture (i):

monatomic so $d = 3$, $N_i = 2N$

$\Rightarrow Q_i = \frac{3}{2}(2N)k(T'-T) = 3Nk(T'-T)$

– final mixture (ii):

diatomic so $d = 5$, $N_f = N$

$\Rightarrow Q_{ii} = \frac{5}{2}Nk(T'-T)$

so $\boxed{Q_{ii} - Q_i = -\frac{1}{2}Nk(T'-T)}$

+2 for $Q = \Delta E_{int}$

−1 if no explanation for why $W = 0$

+1 correct expressions for ΔE_{int} and final answer

Midterm 1, Problem 6 Full solution

6 a) density $\rho = \frac{M}{V}$ so we need to do volume expansion.

method 1

$\Delta V = V_0 \beta \Delta T = V_0 (3\alpha) \Delta T$ since $\beta \approx 3\alpha$

Let V_0 be V at 0°C

Then $V_{40} = V_0 + \Delta V_{0\to 40} = V_0 + V_0(3\alpha)(40-0)$

$= V_0(1 + (3\alpha)(40))$

$\frac{\rho(0)}{\rho(40)} = \frac{M/V_0}{M/V_{40}} = \frac{V_{40}}{V_0}$ since mass doesn't change

$= 1 + (3\alpha)(40) \approx 1.00144$

Note: We could also find V_0, V_{40} in terms of V_{10} but this gives similar answer.

Method 2

Suppose we forget $\beta \approx 3\alpha$.

$\rho = \frac{M}{V} = \frac{M}{\pi r^2 l}$. r_0, l_0 are r, l at 0°C.

$r_{40} = r_0(1 + \alpha(40-0))$

$l_{40} = l_0(1 + \alpha(40-0))$

then $\frac{\rho_0}{\rho_{40}} = \frac{r_{40}^2 l_{40}}{r_0^2 l_0} = (1+\alpha 40)^3 \approx 1 + 3\alpha(40)$

$= 1.00144$

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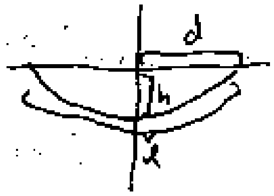
b) 

$d = \frac{[illegible]}{2} = 650$ fixed (half of tower distance)

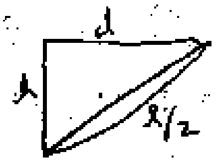
$h = 145$ m @ 10°C (sag)

l = total length of cable

$l_{40} = l_{10}(1 + \alpha(30))$

$l_0 = l_{10}(1 - \alpha(10))$

so $l_{40} - l_0 = l_{10}(\alpha(40))$.

Approximate method to find l_{10}: triangle! 

$l/2 \approx \sqrt{h^2 + d^2} = 666$ m

$l_{10} \approx 1332$ m

then $\Delta l_{0\to 40} = (1332)(40\alpha) = \boxed{0.639\text{ m}}$

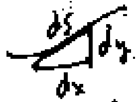
Full (messy) method 

parabola: $y = A x^2$ for A unknown

$h = A d^2 \Rightarrow A = \frac{h}{d^2}$

$$y = \frac{h}{d^2} x^2$$

To find length of a curve $y(x)$



arc length $ds = \sqrt{dx^2 + dy^2}$

We want $\int ds = \int \sqrt{dx^2 + dy^2} = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

In this case $\frac{l}{2} = \int_0^d dx \sqrt{1 + \frac{4h^2}{d^4} x^2}$

Clean up a bit: let $u = \frac{2h}{d^2} x$; $du = \frac{2h}{d^2} dx$

$$\frac{l}{2} = \frac{d^2}{2h} \int_0^{2h/d} du \sqrt{1 + u^2}$$

How do we integrate $\int \sqrt{1+u^2}\, du$?

Method 1

Trig: let $u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$\left(\int du \sqrt{1+u^2} =\right) \int \sec^3\theta \, d\theta = \int \frac{d}{d\theta}(\tan\theta) \sec\theta \, d\theta$

$'' = \sec\theta \tan\theta - \int \tan\theta \sec\theta \tan\theta \, d\theta$

$\int \sec^3\theta \, d\theta = \sec\theta \tan\theta - \int (\sec^2\theta - 1) \sec\theta \, d\theta$

$2 \int \sec^3\theta \, d\theta = \sec\theta \tan\theta + \int \sec\theta \, d\theta$

$= \sec\theta \tan\theta + \ln(\sec\theta + \tan\theta)$

$\int \sec^3\theta \, d\theta = \frac{1}{2}\left(\sec\theta \tan\theta + \ln(\sec\theta + \tan\theta)\right)$

$\Rightarrow \int_0^{2h/d} du \sqrt{1+u^2} = \frac{1}{2}\left(\frac{2h}{d}\right)\sqrt{1 + \left(\frac{2h}{d}\right)^2} - 0$
$+ \frac{1}{2} \ln\left(\sqrt{1 + \left(\frac{2h}{d}\right)^2} + \frac{2h}{d}\right) - 0$

$\Rightarrow l = \frac{d^2}{2h}\left[\left(\frac{2h}{d}\right)\sqrt{1 + \left(\frac{2h}{d}\right)^2} + \ln\left(\sqrt{1 + \left(\frac{2h}{d}\right)^2} + \frac{2h}{d}\right)\right]$

Method 2

Integrate by parts immediately

$\int du \sqrt{1+u^2} = u\sqrt{1+u^2} - \int \frac{u^2}{\sqrt{1+u^2}} du$

$= u\sqrt{1+u^2} - \int \left(\frac{1+u^2}{\sqrt{1+u^2}} - \frac{1}{\sqrt{1+u^2}}\right) du$

$\Rightarrow 2 \int du \sqrt{1+u^2} = u\sqrt{1+u^2} + \int du \frac{1}{\sqrt{1+u^2}}$

$\int du \sqrt{1+u^2} = \frac{1}{2}\left(u\sqrt{1+u^2} + \sinh^{-1}(u)\right)$

$l = \frac{d^2}{2h}\left[\frac{2h}{d}\sqrt{1 + \left(\frac{2h}{d}\right)^2} + \sinh^{-1}\left(\frac{2h}{d}\right)\right]$

Both are same b/c $\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$

In either case

$\underline{\ell = 1342\,m}$. Our triangle approx of 1331 m isn't all that bad.

Then $\boxed{\Delta \ell_{0\to 340} = 0.644\ m}$

also not far off from triangle.

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c) Note first that there is no metal along the sag itself. We cannot just write $\Delta h = h_{10}(40\,\alpha)$ as with the cable. We must be more clever.

Triangle approx: Recall $\frac{\ell}{2} \simeq \sqrt{h^2+d^2}$

so

$$\sqrt{\frac{\ell^2}{4}-d^2} = h$$

Then $h_0 = \sqrt{\frac{\ell_0^2}{4}-d^2}$ $\quad h_{40} = \sqrt{\frac{(\ell_{40})^2}{4}-d^2}$

$\ell_0 = 1331.8$ $\quad \Rightarrow 144.6\ m \quad = 146.2\,m$

$\ell_{40} = 1332.5$

so $\boxed{\Delta h_{0\to 40} = 1.55\,m}$ (should be slightly smaller but dropped some sig figs along the way.)

One could also use calculus:

$$\frac{(d\ell)}{2} = \frac{h\,(dh)}{\sqrt{h^2+d^2}}$$

$$\Rightarrow \frac{\sqrt{h^2+d^2}}{2}(d\ell) = h\,(dh)$$

Note: several people said $\frac{\Delta \ell}{\ell} = \frac{\Delta h}{h}$. This is actually wrong! h & ℓ not proportional.

$$\Rightarrow \tfrac{1}{4}\ell\,(d\ell) = h\,(dh)$$

so $\frac{1}{4}\ell\,\Delta\ell \simeq h\,\Delta h$ if changes small.

$$\Delta h = \tfrac{1}{4}(1332)\,\frac{(0.639)}{145} = \underline{\boxed{1.47\,m}}$$

also good.

Full (ugly) method

Recall: $\ell = \frac{d^2}{2h}\left[\frac{2h}{d}\sqrt{1+\left(\frac{2h}{d}\right)^2} + \sinh^{-1}\left(\frac{2h}{d}\right)\right]$

Let $w \equiv \frac{2h}{d}$ for ~~[illegible]~~ making steps easier to read.

$$\ell = d\left[\sqrt{1+w^2} + \frac{1}{w}\sinh^{-1}(w)\right].$$

Inverting this equation to find $w(l)$ is impossible.

Using calculus as above, (using Δ's to avoid confusion w/ d)

$$\Delta l \cong d\left(\frac{w\,\Delta w}{\sqrt{1+w^2}} - \frac{\Delta w}{w^2}\sinh^{-1}(w) + \frac{1}{w}\frac{1}{\sqrt{1+w^2}}\Delta w \right)$$

$$\cong d\left(\frac{w}{\sqrt{1+w^2}} - \frac{\sinh^{-1}(w)}{w^2} + \frac{1}{w}\frac{1}{\sqrt{1+w^2}}\right)\Delta w$$

Using $w = 2h/d = 0.446$

and $\Delta l = 0.644$ m

we find $\Delta w = 3.52 \times 10^{-3}$

$\Delta w = 2\frac{\Delta h}{d} \Rightarrow \boxed{\Delta h = 1.14\text{m}}$

And we find triangle approx wasn't so good for sag, but still ok.

But do note that Δh is larger than Δl.

The sag not only gets longer b/c the cable is longer but also because the geometry of the cable has changed.

(6). a) General guidelines:

+2 pts for correct ΔV's (with $\beta = 3\alpha$ or some other correct method)

+2 pts for $\frac{\rho_o}{\rho_{Ho}} = \frac{M/V_o}{M/V_{Ho}} = \frac{V_{Ho}}{V_o}$

+1 pt for doing everything else right

Specifics: totals 3 pts if did correctly with $\rho = \frac{M}{L}$ linear density

3 pts if knew that volume was needed but couldn't find it (i.e. didn't know β)

4 pts if got volume partially or did something like $\frac{V_{Ho}}{V_o} = \frac{\Delta V_{Ho}}{\Delta V_o}$, etc.

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b) General guidelines

+2 pts if clearly said $\alpha = 12 \times 10^{-6}$, $\Delta T = 40$ and $\Delta L = \alpha \Delta T (L_0)$ and had to find L_0.

+3 pts: finding L_0

- Method 1: Triangle: gave 2 pts. out of 3

Method 2: Full nasty integral: gave up to 3 pts out of 3

1 pt for correct $y(x)$

1 pt for correct $\int ds$ arc length integral

1 pt for correct (non-calculator) evaluation

Specifics:

- 1 pt if used $\Delta L = \alpha \Delta T L_0$ with $L_0 = 1340, 145$ or something very wrong
- $-\frac{1}{2}$ pt for forgetting a factor of 2 in L_0 equation
- 1 pt for trying it with ρ but got nowhere

c) General guidelines

+ 2pt for clearly stating that the problem is the reverse of part b's method and demonstrating some knowledge of the procedure (or devising an equivalent method)

+ 3pt for actually evaluating it

2/3 pts given for triangle approx

3/3 pts given for full nasty answer

Specifics

- 3 pt total if write down general integral to evaluate and stuff is all in place correctly
- 1 pt if assumed sag was linear

  either $\frac{\Delta sag}{sag} = \frac{\Delta L}{L}$ or $\Delta sag = (sag)_0 \, \alpha \Delta T$
- 2 pt for arguing $sag \propto L^2$ or $\Delta sag \propto \Delta L^2$