1. a) In order for the system to be in equilibrium the net force must be equal to centripital acceleration.

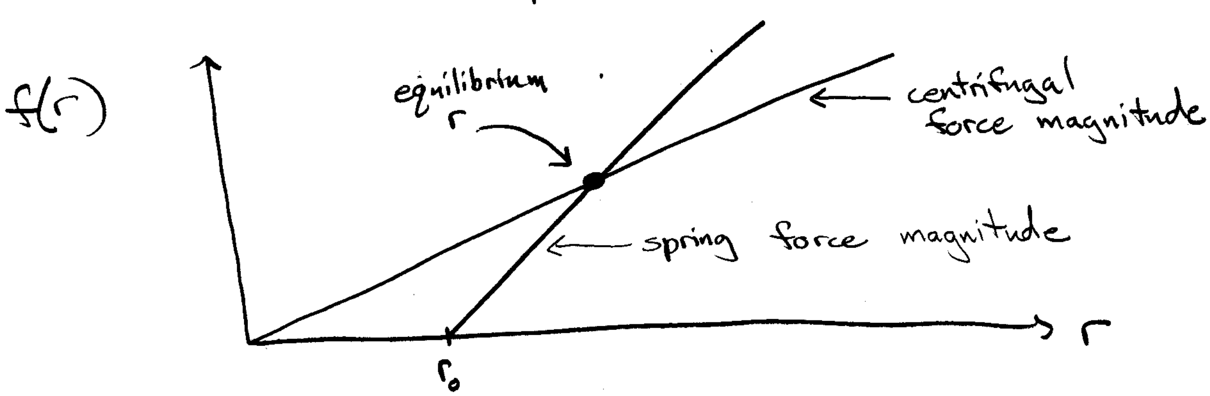
$$\therefore \quad k(r-r_0) = m r \omega^2$$

$$\Rightarrow \quad r = \frac{k r_0}{k - m\omega^2}$$

The equilibrium r increases as ω increases because the denominator goes down. As the denominator goes to zero, however, there is no finite equilibrium value of r.

$$\therefore \quad \boxed{\omega_{max} = \sqrt{\frac{k}{m}}}$$

Another way to think about it is in the rotating reference frame, where you must balance the spring force with the fictitious centrifugal force. We can plot these:



If the slopes of the two forces are the same, then they will never cross.

$$\left.\begin{aligned} |F_{centrifugal}| &= m\omega^2 r \\ |F_{spring}| &= k(r-r_0) \end{aligned}\right\} \quad \text{Set } m\omega^2 = k \quad \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \quad \text{again.}$$

b) Set $r = R$ and invert the equation before to solve:

$$R = \frac{k r_0}{k - m\omega^2} \Rightarrow \boxed{\omega_R^2 = \frac{k}{m}\left(1 - \frac{r_0}{R}\right)} \text{ or } \omega_R^2 = \frac{k(R - r_0)}{Rm}$$

c) Total mechanical energy is the kinetic energy of the system + potential energy of the system.

$$E_{tot} = \frac{1}{2} I_{disk}\,\omega^2 + \frac{1}{2} I_{blocks}\,\omega^2 + k(r - r_0)^2$$

$$= \frac{1}{2}\left(\frac{1}{2} M R^2\right)\omega^2 + \frac{1}{2}\left(2 m r^2\right)\omega^2 + k(r - r_0)^2$$

$$\boxed{= \frac{1}{4} M R^2 \omega^2 + m r^2 \omega^2 + k(r - r_0)^2 \quad \text{with} \quad r(\omega) = \frac{k r_0}{k - m\omega^2}}$$

(If you carry it all the way through your answer will be:

$$E_{tot} = \frac{1}{4} M R^2 \omega^2 + \frac{m k^2 r_0^2 \omega^2}{(k - m\omega^2)^2} + k\left[\frac{m\omega^2 r_0}{k - m\omega^2}\right]^2$$)

Problem 2 Solt'n:

(a) Energy, Angular Momentum (about CM of rod), and Linear Momentum are conserved:

$$\boxed{E_i = \frac{1}{2} m v_0^2 = E_f = \frac{1}{2} m v_f^2 + \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I \omega^2}$$

where $v_f \rightarrow$ final velocity of ball

$V_{cm} \rightarrow$ final velocity of CM of rod

$\omega \rightarrow$ angular velocity of rod about CM

$v_0 \rightarrow$ initial velocity of ball

$$L_{i,\text{ball}} = (\vec{r} \times \vec{P}_i)_{\text{about CM of rod}} = m v_0 h$$

$$L_{f,\text{ball}} = (\vec{r} \times \vec{P}_f)_{\text{about CM of rod}} = m v_f h$$

$$L_{f,\text{rod}} = (I\omega)_{\text{about CM, rod}} = I_{\text{rod}}\, \omega$$

$$\Rightarrow \boxed{m v_0 h = m v_f h + I_{\text{rod}}\, \omega}$$

And $\boxed{P_{i,x} = m v_0 = P_{f,x} = m v_f + M V_{cm}}$

where (+) x is to the right.

So, we have 3 eq'ns w/ 3 unknowns, ω, v_f, and V_{cm}

After some lovely algebra that I'll omit here (& $I_{rod} = \frac{ML^2}{3}$ → remember this is a rod length $2L$)

$$V_{CM_{rod}} = \frac{V_0\, 2mL^2}{(m+M)L^2 + 3mh^2}$$

$$\omega = \frac{24mh}{(m+M)L^2 + 3mh^2}$$

and $$V_f = \frac{V_0\left(1 + \frac{3h^2}{L^2} - \frac{M}{m}\right)}{1 + \frac{3h^2}{L} + \frac{M}{m}}$$

(b) As there are no external torques on the system, we can conserve angular momentum – the question is "about which point?" Now, the most useful point will be the new CM of the system since all other points (like old CM of rod) will be both translating & rotating, so solving for "ω_f" of those points aren't so useful.

→ ok new CM of system: (from CM of rod) is $r = \frac{mh}{M+m}$

If we conserve L about this point, we need new I_{final} about this point:

$$I = \frac{ML^2}{3} + \underbrace{Mr^2}_{\substack{\text{Parallel} \\ \text{Axis } I_{rod}}} + \underbrace{m(h-r)^2}_{\substack{\text{Add point} \\ \text{mass @ distance} \\ (h-r) \text{ from new CM}}} = \frac{ML^2}{3} + \frac{h^2 mM}{(m+M)}$$

So then conserving about new CM:

$$L_i = mV_0(h-r)$$
$$= mV_0\left(\frac{Mh}{M+m}\right)$$

$$L_f = I_{total}\,\omega = \left(\frac{ML^2}{3} + \frac{h^2 mM}{m+M}\right)$$

$$L_i = L_f$$

$$\Rightarrow \boxed{\omega = \frac{3mV_0 h}{L^2(m+M) + 3h^2 m}}$$

Problem 3 Solution

a) $\boxed{W_g = mgL}$

b) $\boxed{W_{sp} = -\frac{1}{2}kL^2}$ $(= -PE_{sp})$

c) $\frac{1}{2}Mv^2 = MgH$ OR $\frac{1}{2}Mv^2 = \frac{1}{2}kL^2 - MgL$

$\Rightarrow \boxed{v = \sqrt{2gH}}$ $\Rightarrow \boxed{v = \sqrt{kL^2 - 2MgL}}$

d) Conservation of momentum (linear):

$$mv_B + M\sqrt{2gH} = (m+M)v_F$$

$$\Rightarrow v_F = \frac{mv_B + M\sqrt{2gH}}{m+M}$$

$$= \frac{\frac{M}{4}v_B + M\sqrt{2gH}}{\frac{M}{4}+M}$$

$$= \frac{\frac{v_B}{4} + \sqrt{2gH}}{5/4}$$

$$v_F = \frac{v_B + 4\sqrt{2gH}}{5}$$

Conservation of energy:

$$\frac{1}{2}\left(\frac{5M}{4}\right)v_F^2 = \frac{1}{2}kL'^2 - \left(\frac{5M}{4}\right)gL'$$

$\left(m+M = \frac{M}{4}+M = \frac{5M}{4}\right)$

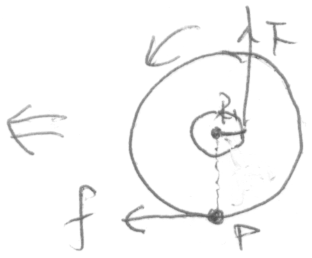
$$\Rightarrow L' = \frac{\left(\frac{5M}{4}\right)g \pm \sqrt{\left(\frac{5M}{4}g\right)^2 + \frac{5M}{4}kv_F^2}}{k}$$

L' must be positive. $\therefore$ ignore the negative solution.

$$\Rightarrow \boxed{L' = \frac{\frac{5M}{4}g + \sqrt{\left(\frac{5M}{4}g\right)^2 + \frac{5M}{4}k\left(\frac{v_B + 4\sqrt{2gH}}{5}\right)^2}}{k}}$$

Prob #4 Solution | Sec 2

(a) For vertical F case:

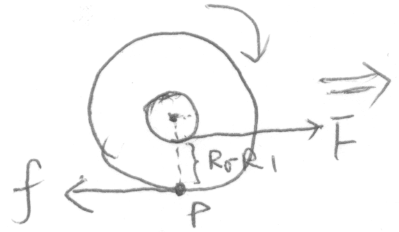


• Spool rotates counter-clockwise (↺) towards the left (←)

∵ Net horizontal force (friction) acting towards left

Net torque ($\tau = F \times R_1$) in counterclockwise direction about the pivot point P.

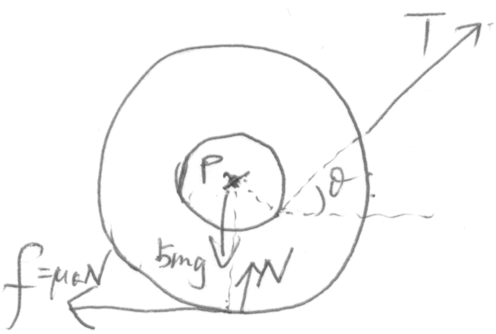
For horizontal F case:



Spool rotates clock-wise (↻) towards right.

∵ Net torque ($\tau = F(R_0 - R_1)$) in clockwise direction, and assume $F > f$

(b)



$$\begin{cases} \sum F_x = T\cos\theta - \mu_k N = 0 & \text{①} \\ \sum F_y = T\sin\theta - 5mg + N = 0 & \text{②} \\ \sum \tau = \mu_k N R_0 - T R_1 = I\alpha = 0 & \text{③} \end{cases}$$

$(a = 0, \ \alpha = 0)$

(c) From ③, $T = \dfrac{\mu_k N R_0}{R_1}$

Substitute it into ①, $\dfrac{\cancel{\mu_k N} R_0}{R_1} \cos\theta_c = \cancel{\mu_k N}$

$$\cos\theta_c = \frac{R_1}{R_0} = \frac{1}{5}$$

$$\sin\theta_c = 2\sqrt{6}/5$$

∴ $\boxed{\theta_c = \arccos(1/5)}$

(d) From ②, $N = 5mg - T_c \sin\theta_c$

Sub. it into ③, $T_c = \dfrac{25\mu_k mg}{1 + 5\mu_k \sin\theta_c} = \dfrac{25\mu_k mg}{1 + 2\sqrt{6}\,\mu_k}$ //

Problem 5 Solution

a) Conservation of angular momentum:

$$L_i = L_f$$

$$\Rightarrow M_0 R_0^2 \omega_0 = \left(M_0 R_0^2 + M_1 R_1^2\right)\omega_f$$

$$= \left(M_0 R_0^2 + \frac{M_0}{2}\frac{R_0^2}{2}\right)\omega_f$$

$$\Rightarrow \boxed{\omega_f = \frac{4}{5}\omega_0}$$

b) Conservation of energy:

$$\frac{1}{2}\left(M_0 R_0^2 + M_1 R_1^2\right)\omega_f^2 = M_1 g\,(2R_1)$$

$$\Rightarrow \frac{1}{2}\left(M_0 R_0^2 + \frac{M_0}{2}\frac{R_0^2}{2}\right)\omega_f^2 = \frac{M_0}{2} g\, \frac{2R_0}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{2}\,\frac{5}{4} M_0 R_0^2 \omega_f^2 = \frac{M_0 g R_0}{\sqrt{2}}$$

$$\Rightarrow \omega_f = \sqrt{\frac{8}{5\sqrt{2}}\,\frac{g}{R_0}}$$

$$\Rightarrow \frac{4}{5}\omega_0 = \sqrt{\frac{8g}{5\sqrt{2}R_0}}$$

$$\Rightarrow \boxed{\omega_0 = \sqrt{\frac{5g}{2\sqrt{2}R_0}}}$$

6a) The weight registered by the scale will go up.

b) If the log is displaced upward by a small amount x, then the buoyant force will decrease by

$$\boxed{\Delta F_{buoyant} = (A_{straw}\, x)\, \rho_L\, g\,.}$$

The original buoyant force would have been cancelled by the weight of the straw, so the expression above is actually the net force acting on the object. We still need to solve for A_{straw}:

At equilibrium: $\underbrace{m_L g}_{\text{weight of straw}} = L A_{straw} \rho_L g$

$$\Rightarrow \boxed{A_{straw} = \frac{m_L}{L \rho_L}}$$

$$\therefore F_{net} = -\frac{m_L \cancel{\rho_L} g}{L \cancel{\rho_L}} x = \boxed{\frac{-m_L g}{L} x}$$

c) The equation of motion will be given by Newton's second law:

$$F_{net} = m \ddot{x}$$

$$\Rightarrow \ddot{x} + \frac{\cancel{m_L}}{\cancel{m_L}} \frac{g}{L} x = 0$$

This is a Harmonic oscillator $\Rightarrow \boxed{\ddot{x} + \frac{g}{L} x = 0}$

$$\omega^2 = \frac{g}{L} \Rightarrow T = \frac{2\pi}{\omega} = \boxed{2\pi \sqrt{\frac{L}{g}}}$$

d) Use Bernoulli's equation and compare the top of the water and just outside the exit hole:

$$P_{atm} + \rho_L g H + \frac{1}{2}\rho v_{top}^2 = P_{atm} + \rho_L g d + \frac{1}{2}\rho v_{hole}^2$$

Two ways to proceed from here:

① Assuming $S_1 \ll S$ then v_{top} will essentially be zero.

$$\therefore \rho_L g(H-d) = \frac{1}{2}\rho_L v_{hole}^2$$

$$\Rightarrow \boxed{v_{hole}^2 = 2g(H-d)}$$

② Making no such assumptions, then $S_1 v_{hole} = S v_{top}$

$$\therefore \rho g(H-d) = \frac{1}{2}\rho_L \left[v_{hole}^2 - \left(\frac{S_1}{S}\right)^2 v_{hole}^2\right]$$

$$\Rightarrow \boxed{v_{hole}^2 = \frac{2g(H-d)}{1-(S_1/S)^2}}$$

e) Fluid will fall a vertical distance d, so it will have time $t = \sqrt{\frac{2d}{g}}$ (from $d = \frac{1}{2}gt^2$) to fall. Then:

$$\boxed{\text{Horizontal distance} = 2\sqrt{d(H-d)}}$$