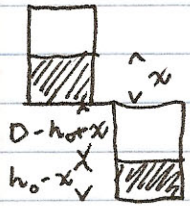
Problem 1, Speliotopoulos Final



The key condition is that the water pressure at the bottom of the left tank must be equal to the gas pressure in the right tank. The water pressure is

$$P_1 = P_0 + \rho g x,$$

and the gas pressure can be determined by

$$\underbrace{\frac{P_0}{\rho_0} = \frac{P_1}{\rho_1}}_{\text{since } P \propto \rho} \rightarrow P_1 = P_0 \cdot \frac{\rho_1}{\rho_0} = P_0 \frac{1/(D-h_0+x)}{1/D}$$

$$= P_0 \frac{D}{D-h_0+x}.$$

Setting the two expressions equal,

$$(P_0 + \rho g x)(D-h_0+x) = P_0 D$$

$$P_0(D-h) + x(P_0 + \rho g(D-h)) + \rho g x^2$$

$$\rightarrow \boxed{x = \frac{-P_0 - \rho g(D-h) + \sqrt{(P_0 + \rho g(D-h))^2 + 4\rho g P_0 h}}{2\rho g}}$$

Any variation in the gas pressure from the top to the bottom is negligible.

② Moments of inertia: $I = \gamma m R^2_{object}$

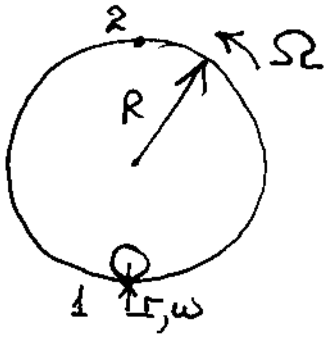
sphere: $\gamma = 2/5$

cylinder: $\gamma = 1/2$

hoop: $\gamma = 1$

a) The center of mass velocity is given by:

No slipping: $v_{cm} = \omega \cdot r$ ← valid at any moment of time

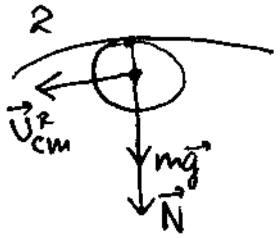


Energy is constant ⇒

$$(\gamma+1)\frac{m\left(v^{(1)}_{cm}\right)^2}{2} = mg\cdot 2R + K_2 \qquad (*)$$

K_2 – kinetic energy at the point 2.

$v^{(1)}_{cm}$ – velocity of CM at point 1.



in the limiting case: $N=0$

$$\Rightarrow \frac{\left(v^{(2)}_{cm}\right)^2}{R} = g \Rightarrow \left(v^{(2)}_{cm}\right)^2 = gR$$

$$\Rightarrow \sqrt{gR} = \left(\omega^{(2)}\right) r \Rightarrow \left(\omega^{(2)}\right) = \frac{\sqrt{gR}}{r}$$

$$\Rightarrow K_2 = \frac{m\left(v^{(2)}_{cm}\right)^2}{2} + \frac{1}{2}\gamma m r^2 \cdot \frac{gR}{r^2} = \frac{mgR}{2} + \frac{\gamma mgR}{2}$$

Plug-in into (*):

$$m\left(v^{(1)}_{cm}\right)^2 = 2mg\cdot 2R + mgR + \gamma mgR \Rightarrow$$

$$(\gamma+1)\left(v^{(1)}_{cm}\right)^2 = (5+\gamma)gR \implies v_{cm}^2 = \left(1 + \frac{4}{\gamma+1}\right)gR$$

max v_{cm} when only one doesn't fall: $\gamma > \frac{1}{2} \Rightarrow v_{cm}^2 < \frac{11}{3}gR$

min v_{cm} when all 3 doesn't fall: $\gamma \le \frac{2}{5} \Rightarrow v_{cm}^2 \ge \frac{27}{7}gR$

Problem 3

v is a velocity of the system after the collision : $v = \frac{v_0}{2}$

$$x = A\sin(\omega t + \phi)$$

If $x = 0$ - the initial position of blocks at $t = 0$ then $\phi = 0$, $\omega = \sqrt{\frac{k}{2m}}$

$$\frac{2m\left(\frac{v_0}{2}\right)^2}{2} = \frac{kA^2}{2}$$

$$A = \sqrt{\frac{mv_0^2}{2k}}$$

$$x = \sqrt{\frac{mv_0^2}{2k}}\sin(\sqrt{\frac{k}{2m}}t)$$

b) to stop the system

$$mv_0 + 2mv = 0$$

$$v = -\frac{v_0}{2}$$

$$v = \dot{x(t)} = -\sqrt{\frac{mv_0^2}{2k}}\sqrt{\frac{k}{2m}}\cos\left(\sqrt{\frac{k}{2m}}T\right) = -\frac{v_0}{2}$$

$$\cos\left(\sqrt{\frac{k}{2m}}T\right) = -1$$

$$\sqrt{\frac{k}{2m}}T = \pi + 2\pi n$$

$$T = (\pi + 2\pi n)\sqrt{\frac{2m}{k}}$$

It also can be easily seen from the following fact:

since the speed should be $\frac{v_0}{2}$ then the point of contact of all blocks should be at $x = 0$ and two blocks should move to the left. The first possible strike is after the half of the period, the second after $\frac{T}{2} + T$ and so on...

Since $T = \frac{2\pi}{\omega}$ then $T_{strike} = (\pi + 2\pi n)\sqrt{\frac{2m}{k}}$

$x(t) = 0$ for $t > T_c$

c) The maximum speed of the system is when there is the maximum initial momentum. The maximum momentum corresponds to the case when two carts are moving to the right (through the $x = 0$). So $T_{strike} = Tn = 2\pi n\sqrt{\frac{2m}{k}}$

$$x(t) = A'\sin\omega' t$$

$$\omega' = \sqrt{\frac{k}{3m}}$$

$$mv_0 + 2m\frac{v_0}{2} = 3mv_f$$

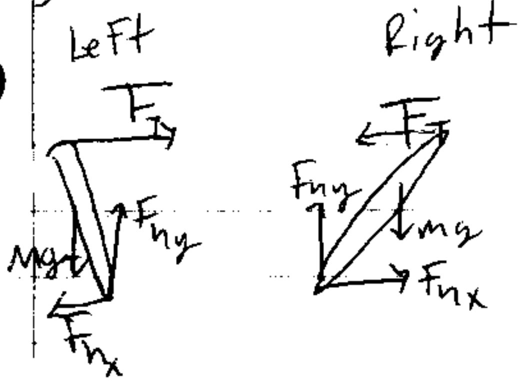
$$v_f = \frac{2}{3}v_0$$

$$\frac{3m(\frac{2}{3}v_0)^2}{2} = \frac{kA'^2}{2}$$

$$A' = 2v_0\sqrt{\frac{m}{3k}}$$

$$x(t) = 2v_0\sqrt{\frac{m}{3k}}\sin\left(\sqrt{\frac{k}{3m}}t\right)$$

4)



on Left → $\Sigma F_x = F_{nx} + F_T$ $T = F_{nx}$

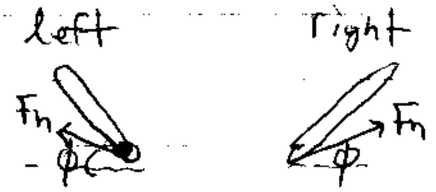
$\Sigma F_y = F_{ny} - Mg = 0$ $F_{ny} = Mg$

$\Sigma \tau_{@\text{hinge}} = Mg\frac{l}{2}\cos\theta - F_T l \sin\theta = 0$

$F_T = \frac{mg}{2}\frac{\cos\theta}{\sin\theta} = \frac{mg}{2}\cot\theta$

$F_{nx} = \frac{mg}{2}\cot\theta$ $F_{ny} = Mg$ $F_n = \sqrt{F_{nx}^2 + F_{ny}^2}$

direction $\phi = \tan^{-1}\left(\frac{F_{ny}}{F_{nx}}\right) = \tan^{-1}(2\tan\theta)$



b) $v = \sqrt{\frac{F_T}{\mu}}$ $f_n = \frac{v}{\lambda_n}$ $\lambda_n = \frac{2L}{n}$ for Lowest f $n=1$

$f = \frac{1}{2L}\cdot\sqrt{\frac{F_T}{\mu}}$ F_T solved in Part a above

Problem 5

$$\gamma\frac{mM}{r_1^2} - T = m\omega^2 r_1$$

$$\gamma\frac{mM}{r_2^2} + T = m\omega^2 r_2$$

$$\gamma\frac{mM}{r_1^3} - \frac{T}{r_1} = \gamma\frac{mM}{r_2^3} + \frac{T}{r_2}$$

$$T = \gamma mM\frac{r_2^3 - r_1^3}{(r_1 + r_2)r_1^2 r_2^2}$$

$$r = \tfrac{r_1+r_2}{2},\ r_1 = r - \tfrac{l}{2},\ r_2 = r + \tfrac{l}{2}$$

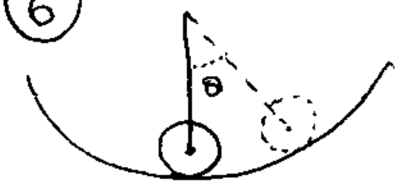
$$T = \gamma mM\frac{r^3\left[(1+\frac{l}{2r})^3 - (1-\frac{l}{2r})^3\right]}{2r^5(1-\frac{l}{2r})^2(1+\frac{l}{2r})^2}$$

$$T = \gamma mM\frac{\left[1+\frac{3l}{2r} - 1 + \frac{3l}{2r}\right](1+2\frac{l}{2r})(1-2\frac{l}{2r})}{2r^2}$$

$$T = \frac{3\gamma mMl}{2r^3}(1 - \frac{l^2}{r^2})$$

$$T = \frac{3\gamma mMl}{2r^3}$$

⑥



a) $\Pi = mgR(1-\cos\theta) = mgR\cdot\frac{\theta^2}{2}$

$$K = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{1}{2}\frac{2}{5}mr^2\omega^2$$

$$v = R\cdot\dot{\theta} = \omega r$$

$$\Rightarrow K = \frac{1}{2}\cdot\frac{7}{5}\cdot mR^2\dot{\theta}^2$$

$\Rightarrow E = K + \Pi = \text{const}$, since there is no slipping

$$\frac{d}{dt}\left(mgR\cdot\frac{\theta^2}{2} + \frac{1}{2}\cdot\frac{7}{5}mR^2\dot{\theta}^2\right) = 0$$

$$\Rightarrow \cancel{m}g\cancel{R}\theta\cdot\cancel{\dot{\theta}} + \frac{7}{5}\cancel{m}R^{\cancel{2}}\cancel{\dot{\theta}}\ddot{\theta} = 0 \Rightarrow$$

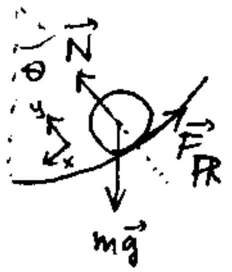
$$\Rightarrow g\cdot\theta + \frac{7}{5}\cdot R\cdot\ddot{\theta} = 0$$

$$\ddot{\theta} + \frac{5}{7}\frac{g}{R}\theta = 0 \Longrightarrow \omega^2 = \frac{5}{7}\frac{g}{R}$$

For a simple pendulum: $\sqrt{\omega^2} = \sqrt{\frac{g}{\ell}}$

$$\Rightarrow \frac{g}{\ell} = \frac{5}{7}\frac{g}{R} \Rightarrow \underline{\underline{\ell = \frac{7}{5}R}}$$

b)



Decomposition gives:

$$y:\ N - mg\cos\theta = 0 \qquad (1)$$

$$x:\ mg\sin\theta - |\vec{F}| = ma \qquad (2)$$

$$\text{torque}:\ \frac{2}{5}mR^2\alpha = F\cdot R \qquad (3)$$

NO SLIPPING: $a = \alpha\cdot R$ (4)

FRICTION FORCE TAKES MAX VALUE WHEN THE BALL IS ABOUT TO SLIP

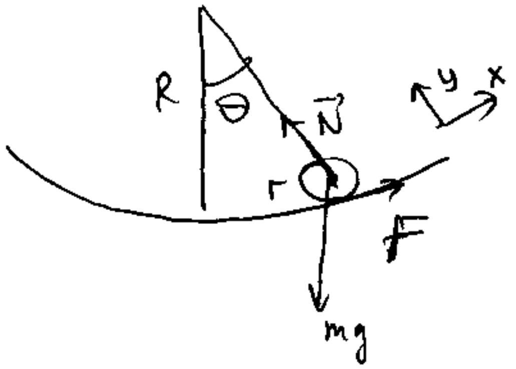
$$\left.\begin{matrix}(3)\\(4)\end{matrix}\right\} \Rightarrow a = \frac{5}{2}\frac{F}{m} \xrightarrow[\text{to (2)}]{\text{Plug-in}} mg\sin\theta = \frac{7}{2}F$$

$$F = \mu_s\cdot N = \mu_s mg\cos\theta_0 \Rightarrow$$

$$mg\sin\theta_0 = \frac{7}{2}\mu_s mg\cos\theta_0$$

Using the condition $\theta_0 \ll 1$ we get $\theta_0 = \frac{7}{2}\mu_s$

⑥ a)



$y: \; N - mg\cos\theta = 0$

$x: \; mg\sin\theta - F = -ma$ (I choose "+a" in positive x direction)

torque: $\frac{2}{5}mr^2\alpha = +F\cdot r$

$-a = \alpha\cdot r$

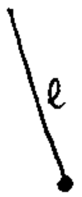
$$\begin{cases} r\alpha = \ddot{\theta}R = a \\ \sin\theta \approx \theta \\ \cos\theta \approx 1 \end{cases}, \quad x = R\cdot\theta - r\cdot\theta \approx R\theta$$

$\Rightarrow \quad F = -\frac{2}{5}ma \quad \Rightarrow \quad mg\sin\theta - F = -ma; \quad F = \frac{-2}{5}ma \Rightarrow$

$g\cdot\theta + \frac{7}{5}R\ddot{\theta} = 0 \quad \Rightarrow \quad \theta + \frac{7}{5}R/g\cdot\ddot{\theta} = 0$

$\Rightarrow \quad \ddot{\theta} + \omega^2\theta = 0$

$\omega^2 = \left(\frac{g}{7/5 R}\right)$



$\Rightarrow \quad \omega^2 = \frac{g}{l} \quad \Rightarrow \quad \underline{\underline{l = \frac{7}{5}R.}}$

Here we assume $R \gg r$

Without that we get $l = \frac{7}{5}(R - r)$