Midterm Examination 2

The two problems given below concern steady plane flows of an inviscid, incompressible fluid, and body forces are neglected. Thus, Euler's hydrodynamical equation reduces to

$$-\boldsymbol{\nabla} p = \rho \boldsymbol{a} = \rho\, D\boldsymbol{V}/Dt \qquad \mathrm{N/m^3}. \qquad (1)$$

1. (a) Let the rectangular Cartesian components of the velocity vector $\boldsymbol{V}$ be expressed in Eulerian and Lagrangian form by

$$u = u_E(x,y) = u_L(x_0,y_0,t)\,, \quad v = v_E(x,y) = v_L(x_0,y_0,t)\,, \quad \mathrm{m/s}\,. \qquad (2)$$

Derive the expressions for the acceleration components Du/Dt and Dv/Dt in terms of the Eulerian velocity components.

(b) Obtain the component form of the vorticity field $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{V} = \boldsymbol{\nabla}\times\boldsymbol{V}$.

(c) Recall that for a material volume element ΔV,

$$\frac{1}{\Delta V}\frac{D(\Delta V)}{Dt} = \mathrm{div}\boldsymbol{V} = \boldsymbol{\nabla}\cdot\boldsymbol{V} \qquad 1/\mathrm{s}\,. \qquad (3)$$

For the incompressible material being considered, write out the condition that results from Eqn. (3).

(d) Give a definition of <u>streamline</u> and show that the streamlines are given by

$$dy/dx = v/u \qquad (u \neq 0)\,. \qquad (4)$$

2. Consider a flow of the form

$$u = -a^2 cy,\ v = b^2 cx, \qquad \mathrm{m/s} \qquad (5)$$

where a, b, and c are <u>positive</u> constants, and a and b are measured in meters.

(a) Verify that this flow is isochoric.

(b) Calculate the vorticity field. Is the flow rotational or irrotational?

(c) Show that the acceleration field is centripetal:

$$\boldsymbol{a} = -d(x\boldsymbol{i} + y\boldsymbol{j})\,, \qquad \mathrm{m/s^2} \qquad (6)$$

where the coefficient d is positive.

(d) If the pressure at the point $(0,0,0)$ is p_0, a constant positive scalar, use Eqn. (1) to calculate the pressure field. Show that the lines of constant pressure are <u>circles</u> and indicate on a sketch the direction in which pressure increases.

(e) Starting with eqn. (4), show that the streamlines for the flow (5) are <u>ellipses</u> and sketch the streamline pattern. Indicate the direction of the velocity field on your sketch.

(f) Using the pressure field you obtained in (d), evaluate the Bernoulli function

$$B(x, y) = p + ½\,\rho V^2. \qquad \mathrm{J/m^3} \qquad (7)$$

(g) Consider the closed streamline C that passes through the points $(a, 0)$ and $(0, b)$. Is $B(x, y)$ constant along C? Is this to be expected for the flow (5)?

(h) Is $B(x, y)$ constant along the straight line $y = x$? Which characteristic property of the flow (5) is related to the behavior you just observed for the Bernoulli function?

(i) Sketch the vortex tube that is embraced by the elliptical streamline C defined in (g).

(j) Recalling that

$$\Gamma(C) = \int_{\mathrm{A}} \mathrm{curl}\,\boldsymbol{V}\cdot\boldsymbol{n}\ \mathrm{dA} = \oint_C \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r}\,, \qquad \mathrm{m^2/s} \qquad (8)$$

calculate the circulation about the circuit C. (Area of ellipse $= \pi\, ab$.)