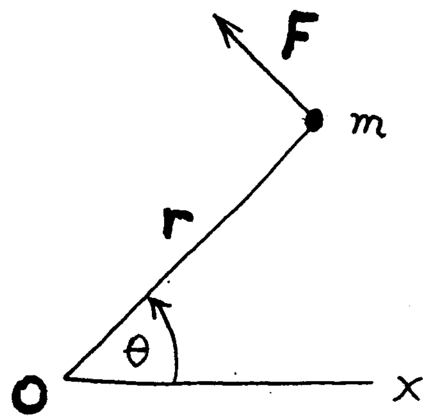
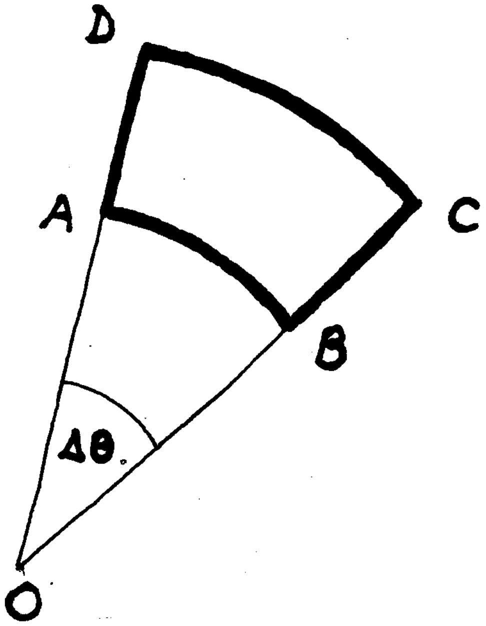
Problem 1 (20 points)





A force $\vec{F}$ exists throughout the plane. The *magnitude* of the force at any point is

$$F = \frac{W_\circ}{r} ,$$

where $W_\circ$ is a constant, and where r is the radial distance from the origin. The *direction* of $\vec{F}$ is tangential and counter-clockwise, as depicted.

(a) A particle of mass m moves in the plane from A to C along the solid line segments shown on the right. It goes by way of point B. Here, A and B are both at distance r_1, while C and D are at distance r_2, where $r_2 > r_1$.

What is W_{ABC}, the work done by the force over the path ABC? $W = \int \vec{F} \cdot d\vec{r}$

$$W_{ABC} = W_{AB} + W_{BC} = -\frac{W_0 r_1 \Delta\theta}{r_1} + 0 = -W_0\theta$$

(b) Now the particle moves from A to C, but going by way of D.

What is W_{ADC}, the work done by the force over the path ADC?

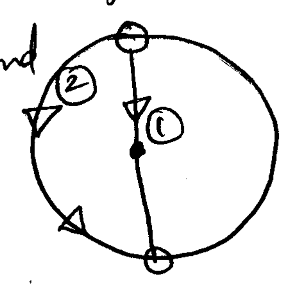
$$W_{ADC} = W_{AD} + W_{DC} = 0 + \frac{-W_0}{r_0}(r_2\Delta\theta) = -W_0\Delta\theta$$

(c) Is $\vec{F}$ a conservative force? Justify your answer.

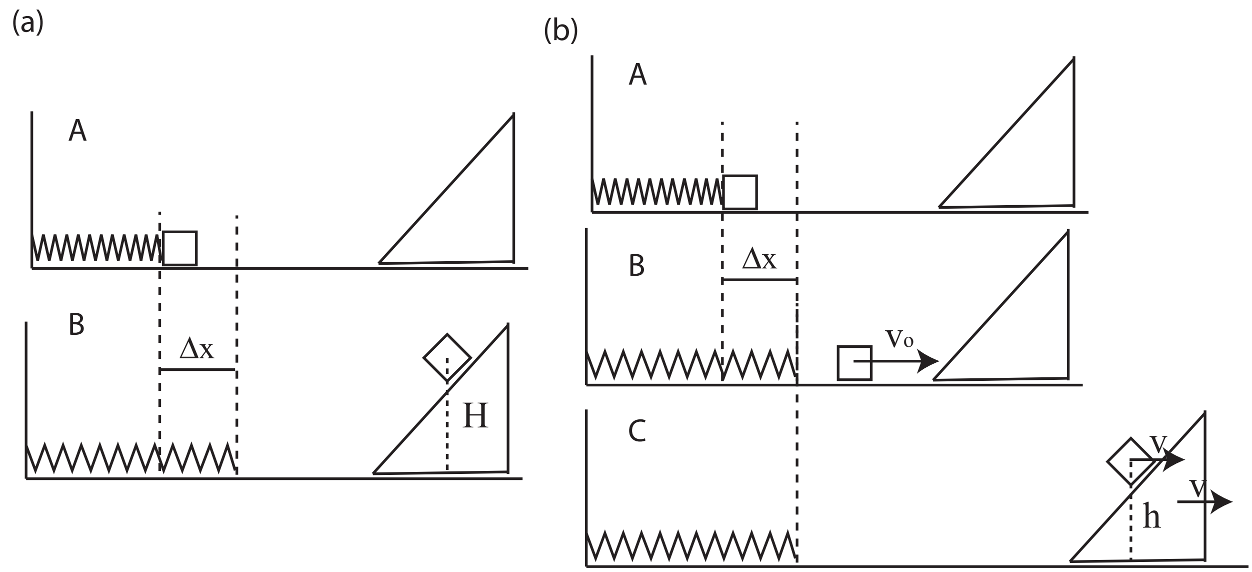
No because consider the following motions

If $\vec{F}$ is conservative, what is the potential energy U at any point?

Work done along ① and ② are different



Problem 2



(a) Total energy is conserved, since there are no dissipative forces (friction).
$E_A = \dfrac{k\Delta x^2}{2}$ and $E_B = mgH$

$$E_A = E_B \Longrightarrow \frac{k\Delta x^2}{2} = mgH \Longrightarrow H = \frac{k\Delta x^2}{2mg}$$

(b) The block keeps going up the incline until the wedge and the block reach the same velocity, call it v. When this happens, the block is stationary relative to the wedge.

Since there are no dissipative forces, the energy of the system throughout the process is conserved: $E_A = E_B = E_C$ (see Fig. 2).

$$E_A = E_C \Longrightarrow \frac{k\Delta x^2}{2} = mgh + \frac{(m+M)v^2}{2} \tag{1}$$

Also, since there are no external forces acting in the horizontal direction after the block stops interacting with the spring, the horizontal momentum of the system is conserved: $p_B = p_C$.

$$mv_o = (m+M)v \Longrightarrow v = \frac{m}{m+M}v_o$$

Substituting v in equation 1, we get:

$$mgh = \frac{k\Delta x^2}{2} - \frac{m^2}{2(m+M)}v_o^2 \tag{2}$$

Now we find v_o, the speed with witch the block leaves the spring, in terms of the known quantities from conservation of energy from A to B:

$$E_A = E_B \Longrightarrow \frac{k\Delta x^2}{2} = \frac{(m)v_o^2}{2}$$

$$v_o^2 = \frac{k\Delta x^2}{m}$$

Substituting v_o in equation 2, we get:

$$mgh = \frac{k\Delta x^2}{2}\left(1 - \frac{m}{m+M}\right) \Longrightarrow h = \frac{k\Delta x^2}{2mg} \cdot \frac{M}{m+M}$$

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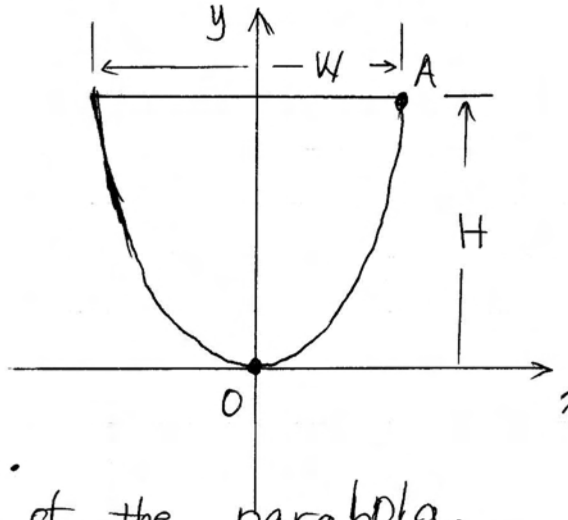


Problem 3.

(a). general expression for a parabola is

$$y = ax^2 + bx + c,$$

where a, b, c are constants.

point $O(0,0)$ is the bottom of the parabola, so,

$$\begin{cases} y = 0 = a\cdot 0^2 + b\cdot 0 + c, & O(0,0) \text{ is on the curve;} \\ \left.\frac{dy}{dx}\right|_{x=0} = 2ax+b\Big|_{x=0} = 2a\cdot 0 + b = 0, & \text{slope at the bottom point is zero.} \end{cases}$$

therefore

$$b=0, \quad c=0,$$
$$y = ax^2$$

because point $A(\frac{W}{2}, H)$ is on the curve, we have

$$H = a\cdot\left(\frac{W}{2}\right)^2, \qquad \boxed{a = \frac{4H}{W^2}}$$

finally, we have the expression

$$\boxed{y = \frac{4H}{W^2}x^2}$$

(b). denote the density of the plate (mass per unit area) as ρ.
then $\rho = \frac{M}{A}$, where A is the area of the plate.
now you have to work out A.

(b.1) calculation of A.

method(i): double integral, integrate over y first, the main point is to find out the correct integration limit.

$$A = \int_{x=-W/2}^{W/2} \int_{y=ax^2}^{H} dx\,dy$$

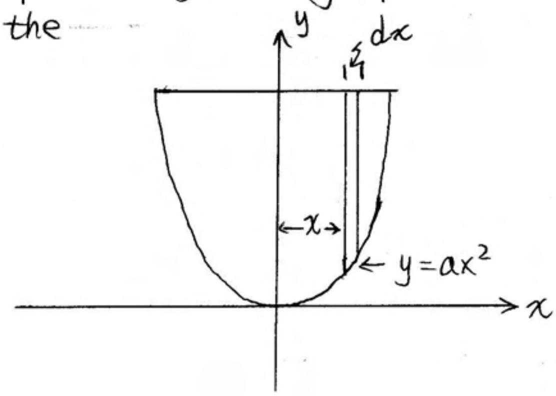
$$= \int_{x=-W/2}^{W/2} (H - ax^2)\,dx$$

$$= \left[Hx - \frac{1}{3}ax^3\right]_{x=-W/2}^{W/2}, \quad \text{use } a \text{ from part (a).}$$

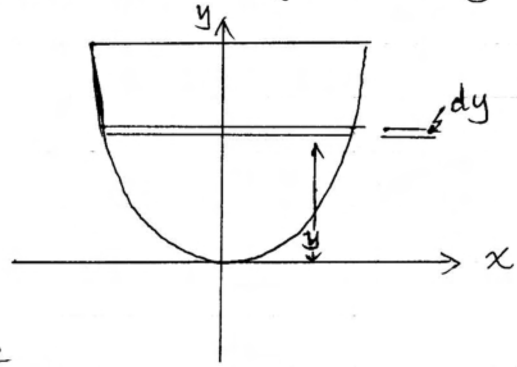
$$= HW - \frac{1}{3} \cdot \frac{4H}{W^2} \cdot \left[\frac{W^3}{8} - \left(-\frac{W^3}{8}\right)\right] = \frac{2}{3}HW.$$

$$\boxed{A = \frac{2}{3}HW}$$

this method is equivalent to slicing the plate in the way given by the ~~picture~~ following picture, and then "sum" all the slices.



method (ii) ; you can slice the plate in a different way.



here it's better to rewrite the expression of the curve as

$$x = \pm\sqrt{y/a}$$

the area of one slice will be

$$dA = 2\sqrt{y/a} \cdot dy,$$

total area is

$$A = \int_{y=0}^{y=H} 2\sqrt{y/a} \cdot dy = 2 \cdot \frac{1}{\sqrt{a}} \cdot \frac{2}{3} y^{\frac{3}{2}} \Big|_0^H$$

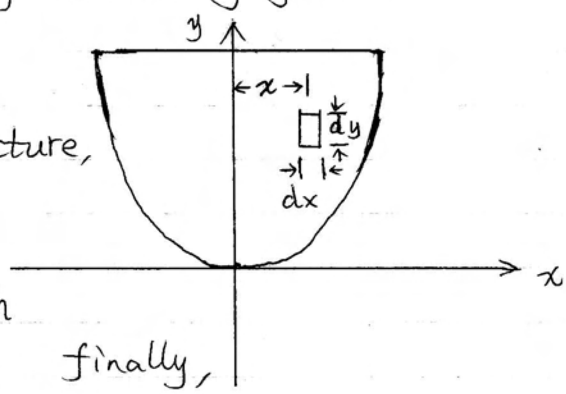
$$= 2 \cdot \frac{1}{\sqrt{4H/W^2}} \cdot \frac{2}{3} \cdot H^{\frac{3}{2}} = \frac{2}{3} HW.$$

again we have,

$$\boxed{A = \frac{2}{3} HW}$$

(b.2) calculation of rotational inertia I.

method (i): double integral, integrate over y first.



by definition, $I = \int r^2 dm$.

for a small patch in the picture,

$$dm = \rho \cdot dx \cdot dy$$

distance of this small patch to the axis is $r = x$, finally,

$$I = \int_{x=-W/2}^{W/2} \int_{y=ax^2}^{H} x^2 \cdot \rho \cdot dx \cdot dy$$

problem 3

$$I = \int_{x=-W/2}^{W} \rho \cdot x^2 \cdot (H - ax^2)\, dx$$

$$= \rho \cdot \left[H \cdot \frac{x^3}{3} - a \cdot \frac{x^5}{5} \right]_{x=-W/2}^{W/2}$$

$$= \rho \cdot \left\{ H \cdot \frac{1}{3} \left[\frac{W^3}{8} - \left(-\frac{W^3}{8}\right) \right] - a \cdot \frac{1}{5} \left[\frac{W^5}{32} - \left(-\frac{W^5}{32}\right) \right] \right\}$$

use a from part (a)

$$= \rho \cdot H \cdot \frac{1}{12} \cdot W^3 - \rho \cdot \frac{4H}{W^2} \cdot \frac{1}{5} \cdot \frac{1}{16} W^5$$

$$= \rho \cdot HW^3 \cdot \left(\frac{1}{12} - \frac{1}{20} \right) =$$

$$= \frac{1}{30} \rho H W^3$$

finally, use $\rho = \frac{M}{A} = \frac{M}{\frac{2}{3}HW}$.

$$I = \frac{1}{30} \cdot \frac{M}{\frac{2}{3}HW} \cdot HW^3 = \frac{1}{20} \cdot M W^2$$

so we have $\boxed{k = \frac{1}{20}}$

$$\boxed{I = \frac{1}{20} M W^2}$$

method (ii')

slicing the plate as in method (ii) of (b.1), and using the formula of rotational inertia for a rod (each slice is a uniform rod rotating around its center), we have,

for a slice at height y,

mass of the slice $= \rho \cdot$ (area of slice)

$= \rho \cdot 2\sqrt{y/a} \cdot dy$

length of the slice $= 2\sqrt{y/a}$

rotational inertia of the slice $dI = \frac{1}{12} \left(2\sqrt{\frac{y}{a}}\right)^2 \cdot \left(\rho \cdot 2\sqrt{\frac{y}{a}}\, dy\right)$

~~total rotational inertia~~

$$I = \int_{y=0}^{y=H} dI = \int_{y=0}^{H} \frac{1}{12} \cdot 4 \cdot \frac{y}{a} \cdot \rho \cdot 2\sqrt{\frac{y}{a}} \cdot dy$$

$$= \frac{2}{3} \rho \cdot a^{-\frac{3}{2}} \cdot \frac{2}{5} y^{\frac{5}{2}} \Big|_{y=0}^{H}$$

$$= \frac{2}{3} \rho \cdot \left(\frac{4H}{W^2}\right)^{-\frac{3}{2}} \cdot \frac{2}{5} \cdot H^{\frac{5}{2}}$$

$$= \frac{1}{30} \rho \cdot H W^3$$

then use $\rho = M/A = M/(\frac{2}{3}HW)$, we have

$$\boxed{I = \frac{1}{30} \cdot \frac{M}{\frac{2}{3}HW} \cdot HW^2 = \frac{1}{20} MW^2}$$

the same answer as method (i).

#4

(a) $E = \frac{1}{2}mv^2 - \frac{GMm}{r} + \underline{\text{const.}}$

$\frac{mv^2}{r} = \frac{GMm}{r^2}$, $\frac{1}{2}mv^2 = \frac{GMm}{2r}$ $\Rightarrow E = -\frac{GMm}{2r} + \underline{\text{const.}}$

(b) $L = mvr \qquad v = \sqrt{\frac{GM}{r}}$

$L = m\sqrt{\frac{GM}{r}}\,r = m\sqrt{GMr}$, $\vec{L}$ point into the page.

(c) Energy way:

$\Delta E = f\Delta s = f\Delta N 2\pi r_0$

$= \Delta\left(-\frac{GMm}{2r}\right) = \frac{GMm}{2r^2}\Delta r$

$\Rightarrow \Delta N = \frac{GMm}{4\pi r_0^2 f}\frac{\Delta r}{r_0}$ #

Torque way:

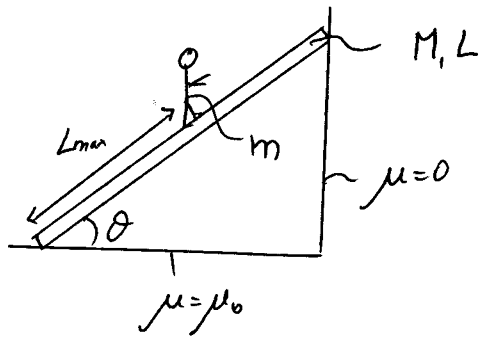
$\tau\Delta t = \Delta L$

$\tau\Delta t = f r_0 \frac{2\pi r_0 \Delta N}{v} = \frac{2\pi r_0^2 f\Delta N}{\sqrt{\frac{GM}{r_0}}} = \frac{2\pi f r_0^{\frac{5}{2}}}{\sqrt{GM}}\Delta N$

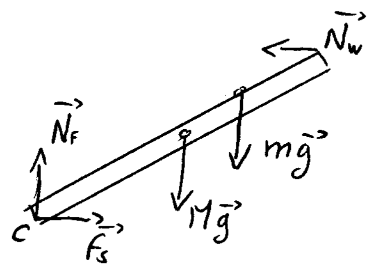
$\Delta L = \Delta(m\sqrt{GMr}) = \frac{1}{2}m\sqrt{GM}\frac{\Delta r}{\sqrt{r_0}}$

$\Rightarrow \Delta N = \frac{\sqrt{GM}}{2\pi f r_0^{\frac{5}{2}}}\frac{1}{2}m\sqrt{GM}\frac{\Delta r}{\sqrt{r_0}} = \frac{GMm}{4\pi f r_0^2}\frac{\Delta r}{r_0}$ #

Problem 5



FBD for the system = ladder



At equilibrium $\quad \sum \vec{F} = \vec{0} \quad$ ①

$\sum \vec{\tau}_c = \vec{0} \quad$ ②

① -> x $\quad f_s - N_W = 0 \qquad$ with $\quad f_s \leqslant \mu_0 N_F \quad$ ③

y $\quad N_F - Mg - mg = 0$

② $\rightarrow \sum \tau_c = -Mg\frac{L}{2}\cos\theta - mg L_{max}\cos\theta + N_W L\sin\theta = 0 \quad$ ④

At the limit before the ladder slips $\quad f_s = \mu_0 N_F$

From ③ $\quad \bullet \; N_F = (M+m)g \quad$ so $\quad f_s = \mu_0 (M+m)g$

$\bullet \; f_s = N_W$

$\Big\} \; N_W = \mu_0 (M+m) g \quad$ ⑤

From ④ $\quad N_W L\sin\theta = \left(\frac{ML}{2} + mL_{max}\right) g\cos\theta$

Together with ⑤ $\quad \mu_0 (M+m) g L\sin\theta = \left(\frac{ML}{2} + mL_{max}\right) g\cos\theta$

$\Rightarrow \quad \mu_0 (M+m) L\tan\theta = \frac{ML}{2} + mL_{max}$

so $\quad \boxed{L_{max} = \mu_0\left(\frac{M}{m} + 1\right) L\tan\theta - \frac{ML}{2m}}$