CS-172 Final May 15, 1993

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Try to keep your answers succinct. The exam is CLOSED BOOK. All questions count equally. First, a few helpful theorems and definitions. Just because a theorem is mentioned, it may not be helpful on the exam.

**Lemma:** The *Pumping Lemma*:

**If** $L$ is regular
**then** $(\exists n)(\forall z \in L$ such that $|z| \geq n)(\exists uvw$ such that $z = uvw$ and $|uv| \leq n$ and $|v| \geq 1)(\forall i) : uv^iw \in L$

**Lemma:** The contrapositive of the *Pumping Lemma*:

**If** $(\forall n)(\exists z \in L$ such that $|z| \geq n)(\forall uvw$ such that $z = uvw$ and $|uv| \leq n$ and $|v| \geq 1)(\exists i) : uv^iw \notin L$
**then** $L$ is not regular.

**Theorem:** *Rice's theorems:* Let $L_{\mathcal{P}}$ be the set of machines with property $\mathcal{P}$. If $\mathcal{P}$ is non-trivial, $L_{\mathcal{P}}$ is undecidable. Further, $L_{\mathcal{P}}$ is r.e. if and only if $\mathcal{P}$ satisfies the following three conditions:

1. If $L \in \mathcal{P}$ and $L \subseteq L'$ for some r.e. $L'$, then $L' \in \mathcal{P}$.
2. If $L$ is an infinite language in $\mathcal{P}$, then there exists a finite subset of $L$ in $\mathcal{P}$.
3. The set of finite languages in $\mathcal{P}$ is *enumerable*.

3-SATISFIABILITY (3SAT)

**INSTANCE:** A boolean formula, $F$, which is an AND of clauses where each clause is an OR of 3 literals.

**QUESTION:** Is $F$ satisfiable?

3-DIMENSIONAL MATCHING (3DM)

**INSTANCE:** A set $M \subset W \times X \times Y$, where $|W| = |X| = |Y| = q$ are disjoint sets.

**QUESTION:** Does $M$ contain a matching, $M' \subset M$, such that no two elements of $M'$ agree in any coordinate.

VERTEX COVER (VC)

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Is there a subset of $K$ vertices which cover all the edges?

CLIQUE

**INSTANCE:** A graph $G$ and integer $K$

**QUESTION:** Does the graph contain a clique (comletely connected subgraph) of $K$ vertices?

HAMILTONIAN CIRCUIT (HC)

**INSTANCE:** A graph $G$

**QUESTION:** Is there a cycle through all the vertices of $G$

PARTITION

**INSTANCE:** A finite set $A$ and a "size" $s(a) \in Z^+$ for each $a \in A$.

**QUESTION:** Is there a subset $A' \subset A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$$

1. Prove or disprove the following languages are regular:
   (a) $L_a = \{a^s b^t : s \geq t \geq 1\}$.
   (b) $L_b = \{a^s b^t : t > s \geq 1\}$. For the proof, use set closure properties and your result about $L_a$. No credit for using the pumping lemma.
   (c) $L_c = \{w : w \text{ contains the substring "0011"}\}$

2. Which of the following are r.e.? Give a proof. (Hint: Any reductions can be done from $L_u$ by creating an $M'$ from $\langle M, w\rangle$ which accepts either $\emptyset$ or $\Sigma^*$ depending on whether $M(w)$ rejects or accepts.)
   (a) $L_{3M} = \{\langle M_1, M_2, M_3\rangle : \text{At least two of the machines accept the same language.}\}$
   (b) $\overline{L_{3M}}$
   (c) $L = \{\langle M\rangle : \ M(\epsilon) \text{ never moves past the } |Q|^{\text{th}} \text{ tape square}\}$. ($Q$ is the set of states of $M$.)

3. Of the following three problems, prove one is in NP, prove one in co-NP, and prove the third is in P.
   (a) **INSTANCE:** Two graphs on the same vertex set $G = (V, E)$ and $H = (V, E')$.
   **QUESTION:** Are $G$ and $H$ **non**-isomorphic?
   (Note that it says "non-isomorphic" rather than "isomorphic".)
   (b) **INSTANCE:** A boolean formula, $F$, on the 100 variables $\{x_1, \ldots, x_{100}\}$.
   **QUESTION:** Is $F$ **un**satisfiable?
   (c) **INSTANCE:** A binary number $n > 1$ in binary.
   **QUESTION:** Is $n$ composite? ("Composite" means "not prime").

4. Prove FEEDBACK VERTEX SET is NP-complete.
   FEEDBACK VERTEX SET

   **INSTANCE:** Directed graph $G = (V, E)$ and integer $K$.

   **QUESTION:** Is there a subset $V' \subset V$ such that $|V'| \leq K$ and every directed circuit in $G$ includes at least one vertex from $V'$.

5. Prove HITTING STRING is NP-complete:

   **INSTANCE:** An integer $n$ and a set of strings $A \subset \{0, 1, \#\}^n$.

   **QUESTION:** Is there a string $x \in \{0, 1\}^n$ such that for each string $a \in A$ there is some $i$, $1 \leq i \leq n$, for which the $i^{\text{th}}$ symbol of $a$ and the $i^{\text{th}}$ symbol of $x$ are identical.

   For example,
   $$A = \{11\#0, 0\#\#\#, \#\#0\#, \#\#\#1, 0\#1\#\}$$
   is a positive instance by choosing $x = 0101$.