ME 170: Engineering Mechanics III
Department of Mechanical Engineering
University of California at Berkeley

Fall Semester 2004

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First Midterm Examination
Thursday October 7 2004
Closed Books and Closed Notes
Answer All Three Questions

Question 1
A Particle Subject to Two Constraints
20 Points

Suppose that the motion of a particle is subject to the following constraints:

$$\begin{aligned} (x\mathbf{E}_2) \cdot \mathbf{v} &= 0, \\ (\mathbf{E}_3) \cdot \mathbf{v} &= 0. \end{aligned} \tag{1}$$

(a) (*5 Points*) Show that one of the constraints (1) is non-integrable. In addition, for the integrable constraint, calculate the function $\psi(\mathbf{r}, t) = 0$.

(b) (*5 Points*) Give a graphical interpretation of the effects of the constraints (1) on the possible motions of a particle. Show in particular that the non-integrable constraint does not place restrictions on where the particle can move but rather how it moves from one location to another.

(c) (*5 Points*) Give prescriptions for the constraint forces associated with the constraints (1).

(d) (*5 Points*) Suppose that, in addition to the constraint forces, a gravitational force $-mg\mathbf{E}_3$ acts on the particle. Using $\mathbf{F} = m\mathbf{a}$, determine the motion of the particle and the constraint forces.

Question 2

A Particle Moving on a Curve

20 Points

Consider a smooth curve which is parameterized by its arc-length parameter s. The position vector of a point on the curve is defined by $\mathbf{r} = \mathbf{r}(s(t))$. Associated with this curve, we define the Serret-Frenet triad $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$:

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds}, \quad \kappa\mathbf{e}_n = \frac{d\mathbf{e}_t}{ds}, \quad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n, \tag{2}$$

where κ is the curvature of the space curve.

(a) (*5 Points*) Given, that the torsion τ and curvature are defined by the relations

$$\kappa\mathbf{e}_n = \frac{d\mathbf{e}_t}{ds}, \qquad \tau\mathbf{e}_n = -\frac{d\mathbf{e}_b}{ds}, \tag{3}$$

show that

$$\frac{d\mathbf{e}_n}{ds} = -\kappa\mathbf{e}_t + \tau\mathbf{e}_b. \tag{4}$$

(b) (*5 Points*) For a particle moving on the space curve, show, with the help of (2), that

$$\dot{\kappa} = v\frac{d\kappa}{ds}, \qquad \mathbf{v} = v\mathbf{e}_t, \qquad \mathbf{a} = \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n, \tag{5}$$

where $v = \dot{s}$.

(c) (*5 Points*) The time-derivative of acceleration, or jerk, is a measure of comfort. Show that the jerk of the particle is

$$\dot{\mathbf{a}} = \left(\ddot{v} - \kappa^2 v^3\right)\mathbf{e}_t + \left(3\kappa v\dot{v} + v^3\frac{d\kappa}{ds}\right)\mathbf{e}_n + \left(v^3\kappa\tau\right)\mathbf{e}_b. \tag{6}$$

Give an example from roller coasters illustrating the role played by $\frac{d\kappa}{ds}$ in this expression.

(d) (*5 Points*) A circular helix has a curvature $\kappa = \frac{1}{R_0(1+\alpha^2)}$ and a torsion $\tau = \kappa\alpha$. For a particle moving on this helix subject to a gravitational force $-mg\mathbf{E}_3$, it can be shown that

$$\dot{v} = -gR_0\tau. \tag{7}$$

Show that the magnitude of the jerk of the moving particle is always non-zero. Using this result, infer that the jerk of a particle moving at constant speed on a circle of radius R_0 is $-\frac{v^3}{R_0^2}\mathbf{e}_t$.

Question 3

A Particle under the Influence of a Conservative Force

35 Points

A particle of mass m is free to move in space and is subject to a single conservative force. The potential energy of the force $\mathbf{P}$ is

$$U = -\frac{\beta}{\|\mathbf{r}\|} - \frac{\gamma}{\|\mathbf{r}\|^3} + \frac{1}{2}K\left(\|\mathbf{r}\| - L\right)^2, \tag{8}$$

where $\beta > 0$, γ, $K > 0$ and $L \geq 0$ are constants. In this expression, the position vector of the particle relative to a fixed origin O is $\mathbf{r}$.

(a) (*5 Points*) With the help of a spherical polar coordinate system, show that

$$\nabla\|\mathbf{r}\| = \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{9}$$

(b) (*5 Points*) Show that the force acting on the particle is

$$\mathbf{P} = -\left(\frac{\beta}{\|\mathbf{r}\|^2} + \frac{3\gamma}{\|\mathbf{r}\|^4} + K\left(\|\mathbf{r}\| - L\right)\right)\frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{10}$$

(c) (*5 Points*) Show that the angular momentum $\mathbf{H}_O$ of the particle can be expressed as

$$\mathbf{H}_O = mR^2\left(\dot{\phi}\mathbf{e}_\theta - \dot{\theta}\sin(\phi)\mathbf{e}_\phi\right). \tag{11}$$

(d) (*5 Points*) With the help of (12), establish the three differential equations governing the motion of the particle.

(e) (*5 Points*) Prove that any solutions to the differential equations you established in **(d)** conserve the total energy E and the angular momentum $\mathbf{H}_O$ of the particle.

(f) (*5 Points*) Given a set of initial conditions $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0)$ for the motion of the particle, why is the particle's motion confined to a plane?

(g) (*5 Points*) Using the results of **(d)** and **(f)**, establish a quintic equation for the radius of a possible circular orbit of the particle about O.

Notes on Spherical Polar Coordinates

Recall that the spherical polar coordinates $\{R, \phi, \theta\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

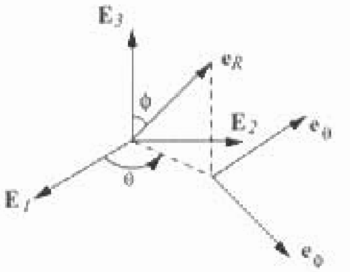


Figure 1: Spherical polar coordinates

For a particle of mass m which is unconstrained, the linear momentum $\mathbf{G}$, kinetic energy T, and acceleration vector $\mathbf{a}$ of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{e}_R + mR\dot{\phi}\mathbf{e}_\phi + mR\sin(\phi)\dot{\theta}\mathbf{e}_\theta, \\ T &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2\right), \\ \mathbf{a} &= (\ddot{R} - R\dot{\phi}^2 - R\sin^2(\phi)\dot{\theta}^2)\mathbf{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi} - R\sin(\phi)\cos(\phi)\dot{\theta}^2)\mathbf{e}_\phi \\ &+ (R\sin(\phi)\ddot{\theta} + 2\dot{R}\dot{\theta}\sin(\phi) + 2R\dot{\theta}\dot{\phi}\cos(\phi))\mathbf{e}_\theta. \end{aligned} \tag{12}$$

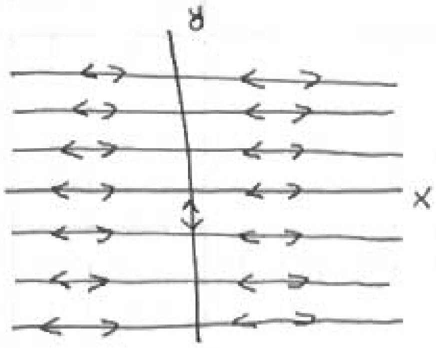
Question 1

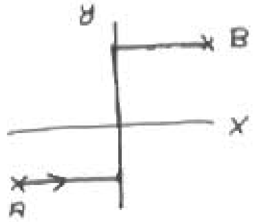
(a) $x\underline{E}_2 \cdot \underline{v} = 0 \Rightarrow x\dot{y} = 0$. Suppose ψ exists such that $\frac{\partial \psi}{\partial y} = x$, $\frac{\partial \psi}{\partial x} = 0$, and $\frac{\partial \psi}{\partial z} = 0$, then $\frac{\partial^2 \psi}{\partial x \partial y} = 1$ but $\frac{\partial^2 \psi}{\partial y \partial x} = 0$. Hence, the constraint $x\underline{E}_2 \cdot \underline{v}$ is non-integrable

The constraint $\underline{E}_3 \cdot \underline{v} = 0$ is equivalent to $\dot{z} = 0$. This equivalent to the integrable constraint $\psi = z - z_0 = 0$, where z_0 is a constant.

(b)

We choose $z_0 = 0$. Hence the particle moves on a plane $z = z_0 = 0$, and its motion is subject to the constraint $x\dot{y} = 0$. Graphically the possible paths of the particle are shown below





Sample path from A to B.

On the $z = 0$ plane, the particle can move on the lines $y = \text{constant}$. The particle can transition to any of these lines by first moving onto the $x = 0$ line (y-axis).

(c)

$$x\underline{E}_2 \cdot \underline{v} = 0 \Rightarrow \underline{F}_c = \lambda_1 x \underline{E}_2$$
$$\underline{E}_3 \cdot \underline{v} = 0 \Rightarrow \underline{F}_c = \lambda_2 \underline{E}_3$$
$$\Bigg\} \quad \underline{F}_c = \lambda_1 x \underline{E}_2 + \lambda_2 \underline{E}_3$$

(d) $\underset{\sim}{F} = m\underline{a}. \Rightarrow$

$$m\ddot{x} = 0$$
$$m\ddot{y} = \lambda_1 x$$
$$m\ddot{z} = \lambda_2 - mg$$

Constraints

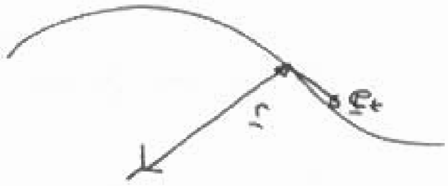
$$z = z_0$$
$$x\dot{y} = 0$$

Soln:

$$x(t) = \dot{x}(0)t + x(0)$$
$$y(t) = y(0) \quad \text{or when } x = 0 \quad y(t) = y(0) + \dot{y}(0)\,t$$
$$z(t) = z_0 = z(0)$$

and $\underline{F}_c = mg\underline{E}_3 + \underline{0}$.

Question 2



(a)

$$\underline{e}_n = \underline{e}_b \times \underline{e}_t$$

$$\frac{d\underline{e}_n}{ds} = \frac{d\underline{e}_b}{ds} \times \underline{e}_t + \underline{e}_b \times \frac{d\underline{e}_t}{ds}$$

$$= -\tau \underline{e}_n \times \underline{e}_t + \underline{e}_b \times \kappa \underline{e}_n = \tau \underline{e}_b - \kappa \underline{e}_t.$$

(b) $\kappa = \kappa(s)$ $\qquad \dot{\kappa} = \frac{d\kappa}{ds}\frac{ds}{dt} = v\frac{d\kappa}{ds}$

$\underline{r} = \underline{r}(s)$ $\qquad \underline{v} = \dot{r} = \frac{dr}{ds}\frac{ds}{dt} = v\underline{e}_t$

$$\dot{\underline{v}} = \dot{v}\underline{e}_t + v\dot{\underline{e}}_t = \dot{v}\underline{e}_t + v\frac{d\underline{e}_t}{ds}\frac{ds}{dt}$$

$$= \dot{v}\underline{e}_t + v^2\frac{d\underline{e}_t}{ds}$$

$$= \dot{v}\underline{e}_t + \kappa v^2 \underline{e}_n$$

(c)

$$\dot{\underline{a}} = \ddot{v}\underline{e}_t + \dot{v}\dot{\underline{e}}_t + \dot{\kappa}v^2\underline{e}_n + 2\kappa v\dot{v}\underline{e}_n + \kappa v^2\dot{\underline{e}}_n$$

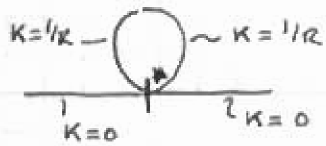
$$= \ddot{v}\underline{e}_t + \kappa\dot{v}v\underline{e}_n + \dot{\kappa}v^2\underline{e}_n + 2\kappa v\dot{v}\underline{e}_n + \kappa v^2 v\frac{d\underline{e}_n}{ds}$$

$$= \ddot{v}\underline{e}_t + 3\kappa v\dot{v}\underline{e}_n + \dot{\kappa}v^2\underline{e}_n + \kappa v^3(\tau\underline{e}_b - \kappa\underline{e}_t)$$

$$= (\ddot{v} - \kappa^2 v^3)\underline{e}_t + \left(3\kappa v\dot{v} + v^3\frac{d\kappa}{ds}\right) + \kappa\tau v^3\underline{e}_b.$$

In roller coasters for a loop-the-loop $\frac{d\kappa}{ds}$ should be a smoothly varying function of s.

For



which is a primitive loop-the-loop $\frac{d\kappa}{ds}$ is not defined at *

(d)

For a particle on a helix: $\dot{v} = -gR_0\tau \Rightarrow v(t) = -gR_0\tau t + v(0)$

$\Rightarrow \ddot{v} = 0$

and $\frac{d\kappa}{ds} = 0$

Hence

$$\dot{\underline{a}} = -\kappa^2 v^3 \underline{e}_t + (3\kappa v(-gR_0\tau))\underline{e}_n + v^3\kappa^2\alpha\,\underline{e}_b$$

As $\kappa > 0$, $\tau > 0$, if the particle is moving $\|\dot{\underline{a}}\| \neq 0$

For a particle on a circle $\kappa = 1/R_0$ and $\tau = 0$

$$\dot{\underline{a}} = -\kappa^2 v^3 \underline{e}_t = -\frac{v^3}{R_0^2}\underline{e}_t.$$

Question 3

(a) $\|\underline{r}\| = R \quad , \quad \underline{r} = R\underline{e}_R$

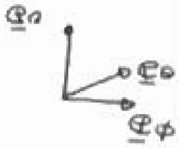
$$\nabla R = \frac{\partial R}{\partial R}\underline{e}_R = \underline{e}_R = \|\underline{r}\|^{-1}\underline{r}$$

(b) $$u = -\frac{\beta}{R} - \frac{\gamma}{R^3} + \frac{1}{2}K(R-L)^2$$

$$\underline{P} = -\frac{\partial u}{\partial R}\underline{e}_R = -\left(\frac{\beta}{R^2} + \frac{3\gamma}{R^4} + K(R-L)\right)\underline{e}_R$$

$$= -\left(\frac{\beta}{\|\underline{r}\|^2} + \frac{3\gamma}{\|\underline{r}\|^4} + K(\|\underline{r}\|-L)\right)\underline{e}_R = P\underline{e}_R$$

(c) $$\underline{H}_0 = \underline{r}\times m\underline{v} = R\underline{e}_R \times m\left(\dot{R}\underline{e}_R + R\dot{\theta}\sin\phi\,\underline{e}_\theta + R\dot{\phi}\underline{e}_\phi\right)$$



$$= mR^2\dot{\theta}\sin\phi\,\underline{e}_R\times\underline{e}_\theta + mR^2\dot{\phi}\,\underline{e}_R\times\underline{e}_\phi$$

$$= mR^2\left(\dot{\phi}\underline{e}_\theta - \dot{\theta}\sin\phi\,\underline{e}_\phi\right)$$

(d)

$\underline{F} = m\underline{a}$

$\cdot\,\underline{e}_R \qquad m\left(\ddot{R} - R\dot{\phi}^2 - R\sin^2\phi\,\dot{\theta}^2\right) = P$

$\cdot\,\underline{e}_\phi \qquad m\left(R\ddot{\phi} + 2\dot{R}\dot{\phi} - R\sin\phi\cos\phi\,\dot{\theta}^2\right) = 0$

$\cdot\,\underline{e}_\theta \qquad m\left(R\sin\phi\,\ddot{\theta} + 2\dot{R}\dot{\theta}\sin\phi + 2R\dot{\theta}\dot{\phi}\cos\phi\right) = 0$

(e) Energy Conservation $\qquad \dot{T} = \underline{F}\cdot\underline{v} = \underline{P}\cdot\underline{v} = -\frac{\partial u}{\partial \underline{r}}\cdot\underline{v} = -\dot{u}$

Define $E = T+U$. Hence $\qquad \dot{E} = \dot{T} + \dot{u} = \dot{T} - \dot{T} = 0 \quad \Rightarrow$ Energy is Conserved

Angular momentum $\qquad \dot{\underline{H}}_0 = \underline{r}\times\underline{F} = \underline{r}\times\underline{P} = R\underline{e}_R \times P\underline{e}_R = \underline{0}$

Hence $\underline{H}_0$ is conserved.

(f) Given $\underline{r}(t_0)$ and $\underline{v}(t_0)$, $\underline{H}_0 = \underline{h}_0$ is defined. As $\underline{h}_0$ is constant $= m\underline{r}\times\underline{v}$

$\underline{r}$ and $\underline{v}$ must lie in a plane containing O. Hence the motion of the particle is planar.

(g) We choose ~~SR~~ such that $\phi = \frac{\pi}{2}$. This is possible because the motion is planar.

In this case the equations of motion simplify to.

$$m(\ddot{R} - R\dot{\theta}^2) = P$$

$$0 = 0$$

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) = 0 \quad \Rightarrow \quad mR^2\dot{\theta} = h \quad (= \underline{H}_o \cdot \underline{E}_3)$$

Hence
$$m\ddot{R} - mR\left(\frac{h}{mR^2}\right)^2 = P$$

and if we substitute for P

$$m\ddot{R} - \frac{h^2}{mR^3} = -\left(\frac{B}{R^2} + \frac{3\gamma}{R^4} + K(R-L)\right)$$

For circular motion $\ddot{R} = 0$. Hence,

$$BR^2 + 3\gamma + K(R-L)R^4 - \frac{h^2R}{m} = 0$$

This is a quintic equation for $R = R_0$, which is the radius of the circular orbit. There may be more than one real positive root to this equation.