CS 172 — Fall 1997	**Final Exam**
Computability and Complexity	**December 15, 1997**

Problem 1. (80 points) Let A be the language of the regular expression $\mathsf{0}^*\mathsf{10} \cup \mathsf{1}^*\mathsf{0}$.

(a) Construct an NFA that accepts A.
(b) Determinize your NFA.
(c) Minimize the resulting DFA.
(d) What is the index of A?
(e) What is the index of the complement of A?

For part (a), you should follow the algorithm for converting a regular expression to an NFA, but you are allowed to take short-cuts that omit ε-transitions. For part (b), use the subset construction. For part (c), use the minimization algorithm.

Problem 2. (40 points) Let A be the language over the alphabet $\{(,),[,]\}$ that contains all balanced strings of parentheses and brackets. For example, $(([]))[] \in A$ and $[) \notin A$.

(a) Give a CFG that generates A.
(b) Give the transition diagram of a PDA that accepts A.

Problem 3. (80 points) For two languages A and B, we define the two languages

$$\mathit{Split}(A,B) = \{x_1 y x_2 \mid x_1 x_2 \in A \text{ and } y \in B\}$$

and

$$\mathit{SymSplit}(A,B) = \{x_1 y x_2 \mid x_1 x_2 \in A \text{ and } y \in B \text{ and } 0 \le |x_1| - |x_2| \le 1\}.$$

For $A = \mathsf{0}^*$ and $B = \mathsf{1}^*$, describe $\mathit{Split}(A,B)$ and $\mathit{SymSplit}(A,B)$ in words. Then prove or disprove each of the following four statements:

(a) If A and B are regular, then $\mathit{Split}(A,B)$ is regular.
(b) If A and B are regular, then $\mathit{Split}(A,B)$ is context-free.
(c) If A and B are regular, then $\mathit{SymSplit}(A,B)$ is regular.
(d) If A and B are regular, then $\mathit{SymSplit}(A,B)$ is context-free.

To prove (a), given finite automata that accept A and B, construct a finite automaton that accepts $\mathit{Split}(A,B)$. To disprove (a), find specific languages A and B for which you can use the pumping lemma for regular languages to show that $\mathit{Split}(A,B)$ is not regular. To prove (b), given finite automata that accept A and B, construct a pushdown automaton that accepts $\mathit{Split}(A,B)$. To disprove (b), find specific languages A and B for which you can use the pumping lemma for context-free languages to show that $\mathit{Split}(A,B)$ is not context-free.

Problem 4. (40 points) Consider the following three languages:

$A_1 = \{\langle M \rangle \mid M$ is a Turing machine that halts on at least one input $\}$
$A_2 = \{\langle M \rangle \mid M$ is a Turing machine that halts on at most one input $\}$
$A_3 = \{\langle M \rangle \mid M$ is a Turing machine that halts on exactly one input $\}$

Which of these languages are recursive, which are r.e., which are co-r.e., and which are neither? You need to justify your answers briefly. You may assume that TMMEMBERSHIP is r.e. but not recursive, TMEMPTINESS is co-r.e. but not recursive, and TMUNIVERSALITY is neither r.e. nor co-r.e.

Problem 5. (80 points) Let f be a monotonically increasing computable function from N to N; that is, $f(n) < f(n+1)$ for all natural numbers $n \in \mathsf{N}$. Let $range(f) = \{y \mid (\exists x) f(x) = y\}$. Prove or disprove each of the following two statements:

(a) $range(f)$ is recursive.
(b) $range(f)$ is r.e.

Let g be any computable function, from Σ^* to Σ^* for some alphabet Σ. Prove or disprove each of the following two statements:

(c) $range(g)$ is recursive.
(d) $range(g)$ is r.e.

Problem 6. (80 points) A linear inequality has the form

$$a_0 x_0 + a_1 x_1 + \cdots + a_n x_n \leq b$$

or

$$a_0 x_0 + a_1 x_1 + \cdots + a_n x_n \geq b,$$

where $a_0, \ldots, a_n, b$ are integer constants, and $x_0, \ldots, x_n$ are variables. A linear formula combines linear inequalities using the boolean operations of AND, OR, and NOT. A linear formula is $\{0,1\}$-satisfiable if the formula can be made true by assigning to each variable either 0 or 1. For example, the linear formula

$$(3x_0 + 2x_1 \leq 1 \vee \ -2x_0 + x_1 \geq 1) \ \wedge \ x_0 \leq 0$$

is $\{0,1\}$-satisfiable (take $x_0 = 0$ and $x_1 = 1$); the linear formula

$$3x_0 + 2x_1 \leq 2 \ \wedge \ x_0 \geq 1$$

is not $\{0,1\}$-satisfiable. For each of the following three problems, either prove that the problem is in P, or that it is NP-complete, or that it is not in NP:

(a) Given a disjunction of linear inequalities, is it $\{0,1\}$-satisfiable?
(b) Given a conjunction of linear inequalities, is it $\{0,1\}$-satisfiable?
(c) Given a linear formula, is it $\{0,1\}$-satisfiable?

You need to justify your answers. You may assume that 3SAT, CLIQUE, HAMPATH, and SUBSET-SUM are NP-complete.