# CS 70 Discrete Mathematics for CS
# Fall 2001 Wagner Midterm 1

PRINT your name: ________________________________________

SIGN your name; ________________________________________

This exam is closed-book, closed-notes. One page of notes is permitted. Calculators are permitted. Do all your work on the pages of this examination.

You have 2 hours. There are 4 questions, of varying credit (50 points total). You should be able to finish all the questions, so avoid spending too long on any one question.

**1. (12 pts.) Short-answer questions**

Translate each of the following claims into symbolic form. For instance, a good translation of “$n$ is either at least three or at most five” would be “$n \geq 3 \quad n \sqcap 5$.”

Then, state whether the claim is true or false, and briefly justify your answer.

(a) [3 pts.] There is some natural number whose square root is not a natural number.

(b) [4 pts.] For every natural number $n$, one can find another natural number $m$ that is strictly smaller than $n$.

(c) [5 pts.] For each natural number $k$ there is some lower bound $\ell$ so that $k^n \geq n!$ when $n \geq \ell$.

**2. (12 pts.) Reachability**

In chess, a bishop can move diagonally in any of the four directions. Consider a $3 \times 3$ board, with a bishop initially placed at the location marked ‘B’ (see below). Prove that it can never reach the square marked ‘X’.

| | | |
|---|---|---|
| | | |
| B | | |
| | | X |

**3. (16 pts.) Proof by induction**

Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 2 \text{ and } a_0 = 1, a_1 = 2.$$

Consider the following argument:

**Theorem 1** $a_n \leq n + 2$ *for all* $n \geq 0$.

**Proof:** We use strong induction on *n*. The base cases $n = 0$ and $n = 1$ hold, since $a_0 = 1 \leq 0 + 2$ and $a_1 = 2 \leq 1 + 2$. Now if $a_i \leq i + 2$ for each $i = 0, 1, \ldots, n-1$, for some $n \geq 2$, then we have

$$a_n = 2a_{n-1} - a_{n-2} \leq 2\big((n-1)+2\big) - \big((n-2)+2\big) = 2n + 2 - n = n + 2,$$

which shows that $a_n \leq n + 2$ holds for all $n \geq 0$. □

(a) [6 pts.] Critique the above proof.

(b) [10 pts.] Give a better proof of the theorem.

**4. (10 pts.) Matchings**

Recall that a *matching* on $n$ boys and $m$ girls is a pairing where each boy is married to exactly one girl and each girl is married to exactly one boy.

(c) [5 pts.] Let $M$ be a stable matching on $n$ boys and $n$ girls where Alice is paired with Bob. Now Alice and Bob fly off the Bermuda on vacation. We are left with a matching, call it $L$, on the remaining $n$–1 boys and $n$–1 girls according to who is still paired up. Is $L$ guaranteed to be a *stable* matching, if $M$ is stable? Prove your answer.

(d) [5 pts.] If $M, M'$ are two matchings, let $M \wedge M'$ denote the configuration where each girl is married to the better of her two partners in $M$ and $M'$ (according to that girl's preference list). Is $M \wedge M'$ guaranteed to be a matching? Prove your answer.
(Note that none of the matchings here are required to be stable.)

**Finished!** You're done; this is the last page; there are no more questions.