CS-174   Exam   April 28, 1992
David Wolfe

   There are 4 easier problems marked (**E**) , and 3 harder problems (**H**) . Each question is 10 points, but your highest score on a hard question is tripled. 60 points is enough for an A on the exam, so a student who gets all the (**E**) questions and one of the (**H**) questions has an A with 10 points to spare.

1. (**E**) How many ways are there to score exactly 60 points on this exam if no partial credit is given on each question? (Note that you might get more than one hard question right — Be careful not to overcount.)

2. (**E**) In the children's game of *buzz*, players count from 1 to 100. But if a number is either a multiple of 7 or has a 7 as one of its digits, the player says "buzz" rather than the number. How many times should "buzz" be said in place of a number in the game? (Use inclusion-exclusion – do not simply enumerate all numbers.)

3. (**E**) What are all winning moves from a nim position consisting of three piles having sizes 5, 9 and 10. Explain.

4. (**E**) Given integers $a_0, a_1, \ldots, a_{100}$, show that there is a consecutive sequence $a_i, \ldots, a_j$, $0 \le i \le j \le 100$, whose sum is a multiple of 100.

5. (**H**) Graph G has *diameter 2* if there is a path of length 2 between any two vertices in G. Let G be a random graph on n vertices with each edge present with probability $p > 0$.

   (a) Find the probability that two vertices, say i and j, are both adjacent to vertex k. (i, j and k are distinct.)

   (b) Find the probability that two vertices, say i and j, have no neighbors in common.

   (c) Lower bound the probability that G has diameter 2.

   (d) When p is fixed, show that this probability tends to 1 as $n \to \infty$.

6. (**H**) Prove the MEX rule. In particular, suppose you are given a green hackenbush position, G, which you've concluded has value $*n$ by using the MEX rule to recursively evaluate all subpositions. Let H be a nim pile of size n and show that the second player wins the game $G + H$ (the game consisting of G and H played side by side.)

7. (**H**) Let $\{A_1, A_2, \ldots, A_{mn}\}$ be mn events each having some known probability $\mathbf{P}\{A_i\} = p_i$. Further, for all j, $\{A_{jm+1}, \ldots A_{(j+1)m}\}$ are mutually independent. So $\{A_1, \ldots, A_m\}$ are independent, as are $\{A_{m+1}, \ldots, A_{2m}\}$, etc.

   (a) Upper and lower bound $\mathbf{P}\{A_1 \vee A_2 \vee \ldots \vee A_{mn}\}$.

   (b) Let N be the number of A_i's which are true. Upper and lower bound $\mathbf{E}\{N\}$.