CS-174 | Final | May 20, 1993

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Limits:

$$(1+1/n)^n \overset{n\to\infty}{\longrightarrow} e \qquad (1-1/n)^n \overset{n\to\infty}{\longrightarrow} 1/e$$

Basic combinatoric identities:

$$\binom{n}{k} \overset{\text{def}}{=} \frac{n!}{(n-k)!k!} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$H_n \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{1}{i} \qquad \frac{1}{2}\lg n \le H_n \le \lg n \quad \text{(Harmonic numbers, coupon collector)}$$

$$n! \approx \sqrt{2\pi n}(n/e)^n \qquad \text{(Stirling's approximation)}$$

Inclusion/exclusion formula:

$$\mathbf{P}\left\{\bigcup_{1\le i\le n} A_i\right\} = \sum_{1\le i\le n} \mathbf{P}\{A_i\} - \sum_{1\le i<j\le n} \mathbf{P}\{A_i \cap A_j\} + \sum_{1\le i<j<k\le n} \mathbf{P}\{A_i\cap A_j \cap A_k\} + \cdots + (-1)^{n+1}\mathbf{P}\{A_1\cap A_2\cap\cdots\cap A_n\}$$

Some probability:

- $\mathbf{P}\left\{\bigvee_i A_i\right\} \le \sum_i \mathbf{P}\{A_i\}$. Equality holds if and only if the events $\{A_i\}$ are disjoint.
- Events A and B are independent if $\mathbf{P}\{A\wedge B\} = \mathbf{P}\{A\}\cdot\mathbf{P}\{B\}$. Random variables X and Y are independent if for all numbers x and y, the events "$X=x$" and "$Y=y$" are independent.
- Events $\{A_i\}$ are pairwise independent if $\forall i,j : \mathbf{P}\{A_i\wedge A_j\} = \mathbf{P}\{A_i\}\mathbf{P}\{A_j\}$. Events $\{A_i\}$ are mutually independent if $\forall S : \mathbf{P}\left\{\bigwedge_{i\in S} A_i\right\} = \prod_{i\in S}\mathbf{P}\{A_i\}$.
- Definition of conditional probability: $\mathbf{P}\{A \mid B\} = \mathbf{P}\{A\wedge B\}/\mathbf{P}\{B\}$.
- For RV's with finite expectation, $\mathbf{E}\{X\} = \sum_i i\mathbf{P}\{X=i\}$ and $\mathbf{E}\{X+Y\} = \mathbf{E}\{X\}+\mathbf{E}\{Y\}$.

Binomial distribution: Let X_n be the number of heads in n flips of a coin with bias p. $\mathbf{E}\{X_n\} = pn$ and

$$\mathbf{P}\{X_n = k\} = \binom{n}{k} p^k(1-p)^{n-k},\ 0\le k\le n$$

Geometric distribution: Let X be the number of tails in a row before getting a heads in a sequence of flips of a coin with bias p. Let $Y = X+1$ be the total number of flips to get a head. Depending on the literature, either X or Y are said to have the geometric distribution.

$$\begin{aligned} \mathbf{P}\{X=k\} &= p(1-p)^k,\ k\ge 0 \\ \mathbf{P}\{Y=k\} &= p(1-p)^{k-1},\ k\ge 1 \end{aligned}$$

$\mathbf{E}\{X\} = \frac{1-p}{p}$ and $\mathbf{E}\{Y\} = \frac{1}{p}$

Markov's Inequality: For a non-negative random variable X,

$$\mathbf{P}\{X\ge\alpha\} \le \frac{1}{\alpha}\mathbf{E}\{X\}$$

König Egerváry Theorem: In a bipartite graph, the size of the maximum matching equals the size of the minimum vertex cover.

Dilworth's Theorem: The minimum number of chains to cover all the elements equals the largest sized antichain.

Euler's Theorem: An undirected, connected multigraph, G, has an Eulerian walk **iff** either none or exactly two of its nodes have odd degree. A strongly connected directed multigraph has an Eulerian cycle **iff** every node has indegree equal to its outdegree.

Sperner's Lemma: Let T be a triangulated triangle. Label the three corners, v_0, v_1, v_2 of T with 0,1 and 2. Label the remaining nodes of T in any way such that

1. Each node along the line connecting v_i and v_j is labeled with either i or j.
2. Each interior node is labeled either 0, 1, or 2.

Then, there exists an interior triangle labeled with 0, 1 and 2.

Euler's Formula: Any planar embedding of a connected graph has $|E|-|V|+2$ regions. The following are corollaries:

1. For any planar graph, $|E| \leq 3|V| - 6$
2. Any planar graph has a vertex, v, of degree $d(v) \leq 5$
3. Any planar graph is 5-colorable. (Also, any planar graph happens to be 4 colorable.)

Also, 3-colorability of planar graphs is NP-complete. Any graph is 2-colorable if and only if it's bipartite or, equivalently, has no odd cycles.

Kuratowski's Th: A graph is planar if and only if it does not contain a graph homeomorphic to $K_{3,3}$ or K_5.

Hall's Theorem: Let G be a graph on vertex sets L on the left and R on the right. G has a perfect matching if and only if

$$\forall A \subseteq L: \ |\Gamma(A)| \geq |A|$$

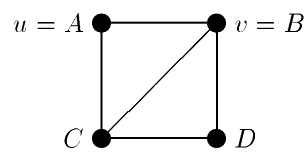
Cayley's Theorem: The number of labeled trees on n vertices is exactly n^{n-2}.

You need not show your work for the first two problems. These first two pages can be removed — do not submit them with your final. The only change from what was handed out is the first section **Limits**. Make your solutions clear — especially if they have appeared in the course before. For example, if you use linearity of expectation, be sure to define the random variable you're adding up. Challenging extra credit problems are available if you hand in your exam early.

1. Let $\{A_1, A_2, \ldots, A_{mn}\}$ be mn events each having some known probability $\mathbf{P}\{A_i\} = p_i$. Further, for all j, $\{A_{jm+1}, \ldots A_{(j+1)m}\}$ are mutually independent. So $\{A_1, \ldots, A_m\}$ are independent, as are $\{A_{m+1}, \ldots, A_{2m}\}$, etc.
   (a) Upper and lower bound $\mathbf{P}\{A_1 \vee A_2 \vee \ldots \vee A_{mn}\}$.
   (b) Let N be the number of A_i's which are true. Upper and lower bound $\mathbf{E}\{N\}$.

2. Five people playing poker are dealt five cards each.
   (a) What is the chance that Bob received a flush (all five cards are the same suit)?
   (b) Let F denote the event, "at least one of the five people receives a flush." Use the first two terms of inclusion-exclusion to give upper and lower bounds on $\mathbf{P}\{F\}$. You need not simplify your answers.

3. Consider a graph G consisting of a set of n vertices, where each edge is present independently with probability $p = 1-q$. In other words, G is a complete graph, K_n, except each edge gets removed with probability $q = 1-p$. Let N be the number of simple paths between a specified pair of vertices u and v. The following example shows a graph where $n = 4$ and $N = 3$; the simple paths between nodes A and B are AB, ACB, and $ACDB$.



   Compute $\mathbf{E}\{N\}$ in terms of n and p. (Do not simplify your result.)

4. Show that a planar graph G with 8 vertices and 13 edges cannot be 2-colored. (Hint: Use facts you know about the regions to prove G must contain a triangle. Do not try to provide an exhaustive search.)

5. You've just discovered incredibly simple linear time randomized algorithms to do the following two problems:
   - Find a maximum sized matching in a bipartite $n \times n$ graph with probability $\frac{1}{2}$
   - Find a minimum sized vertex cover in a bipartite $n \times n$ graph with probability $\frac{1}{2}$

   The only problem is that when, for example, the first algorithm fails, it prints out a matching that is not optimal, and there's no way for you to test for optimality quickly.

   If you were to rerun the algorithm 10 times, and then take the maximum sized matching found so far, with probability $(1 - 2^{-10})$ you'll have the maximum matching. But you wish to write an algorithm which is **guaranteed** to always give a maximum matching. (You're willing to accept that the algorithm takes an arbitrarily long time with vanishingly small probability.)

   (a) Design an algorithm which always gives a maximum matching using the "•" algorithms as subroutines. (Feel free to give up a little bit in efficiency to make part (b) below easy, as long as the expected running time is linear.)
   (b) Find the expected running time of your algorithm. Assume the running time of the two routines is cn. (1 point off if answer is not simplified.)

6. In class we showed that a complete binary tree of height h cannot be embedded in a hypercube of dimension $h+1$. Show that *any* tree T can be embedded in some n-cube. (You may make n as large as you wish.)