ME 104: ENGINEERING DYNAMICS II
Department of Mechanical Engineering
University of California at Berkeley

Fall Semester 2007

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Second Midterm Examination
Wednesday November 7 2007
Closed Books and Closed Notes

Question 1
A Single Particle (20 Points)

As shown in Figure 1, a particle of mass m is in motion about a fixed point O. A force $\mathbf{F}$ acts on the particle. This force is a central force, i.e., $\mathbf{F} \parallel \mathbf{r}$.

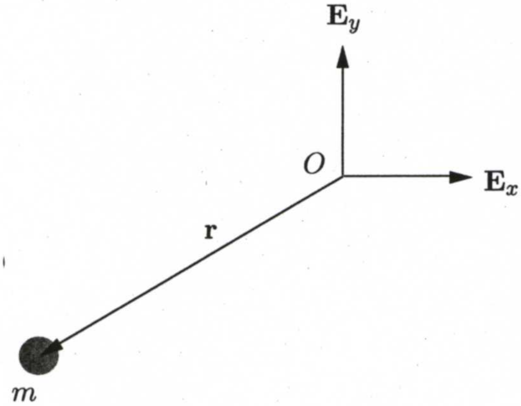


Figure 1: A particle moving on a planar path under the influence of a central force $\mathbf{F}$.

(a) *(6 Points)* Starting from the representation $\mathbf{r} = r\mathbf{e}_r$, establish expressions for the kinetic energy T and angular momentum $\mathbf{H}_O$ of the particle.

(b) *(4 Points)* Starting from the angular momentum theorem for a single particle, prove that $mr^2\dot{\theta}$ is conserved.

(c) *(5 Points)* Starting from the work-energy theorem for a single particle $\dot{T} = \mathbf{F} \cdot \mathbf{v}$, prove that the total energy of the particle is conserved if

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r. \tag{1}$$

In your solution, give a clear expression for the total energy E of the particle.

(d) *(5 Points)* A satellite is in motion in an elliptical orbit about the Earth. Show that the radial velocity $v = \dot{r}$ of the satellite varies as its distance r from the Earth:

$$v^2 = \frac{2E_0}{m} + \frac{2GM}{r} - \frac{h^2}{m^2r^2}, \tag{2}$$

where E_0 and h are constants and M is the mass of the Earth.

Question 2

A System of Two Particles (35 Points)

As shown in Figure 2, a satellite of mass m_1 is connected to a spacecraft of mass m_2 by a tether of length r. By varying the length of the tether, the rotation of the spacecraft-satellite about their center of mass C can be changed and this can then be used to artificially induce gravity.

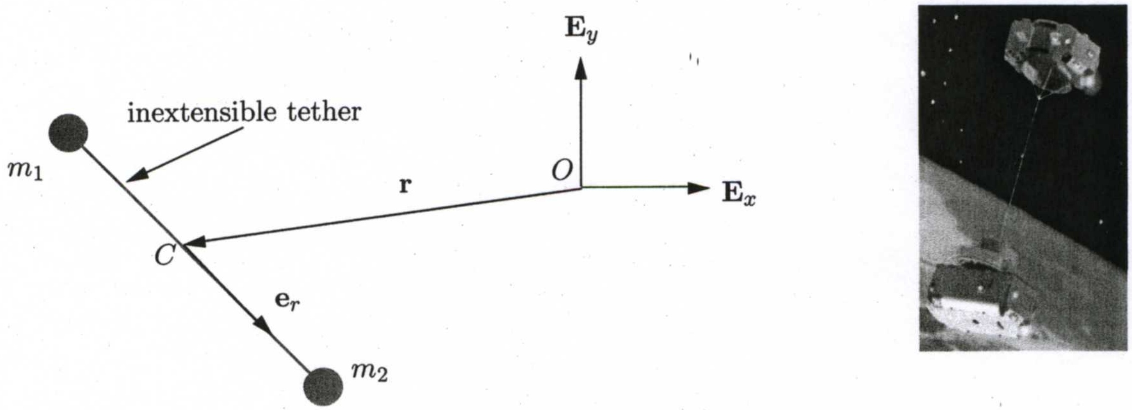


Figure 2: Schematics of a satellite and spacecraft connnected by an inextensible tether of length $r = r(t)$.

(a) *(8 Points)* Starting with the representations for the position vector $\mathbf{r}$ of the center of mass C and the position vectors of the particles

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \quad \mathbf{r}_1 = \mathbf{r} - r_1\mathbf{e}_r, \quad \mathbf{r}_2 = \mathbf{r} + r_2\mathbf{e}_r, \tag{3}$$

where $r = r_1 + r_2$, show that

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r, \quad r_2 = \left(\frac{m_1}{m_2}\right) r_1, \quad m_1 r_1^2 + m_2 r_2^2 = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2. \tag{4}$$

(b) *(10 Points)* Show that the linear momentum and angular momentum of the system are

$$\mathbf{G} = (m_1 + m_2)\,\dot{\mathbf{r}}, \quad \mathbf{H}_O = \left((m_1 + m_2)\,(x\dot{y} - y\dot{x}) + \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \dot{\theta}\right) \mathbf{E}_z. \tag{5}$$

(c) *(4 Points)* Draw free-body diagrams for each of the particles. Give a clear expression for the tension force in the tether.

(d) *(4 Points)* Show that the kinetic energy T and the linear momentum $\mathbf{G}$ of the system are constant. Clearly indicate any intermediate results that you use.

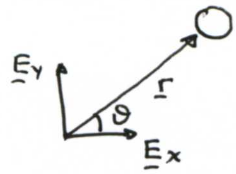
(e) *(4 Points)* Using the angular momentum theorem $\dot{\mathbf{H}}_C = (\mathbf{r}_1 - \mathbf{r}) \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}) \times \mathbf{F}_2$, show that

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \dot{\theta} = h, \tag{6}$$

where h is a constant.

(f) *(5 Points)* Suppose an acceleration of $0.75g$ for an object on the satellite is sought. Show that this can be achieved if $\left(\frac{m_2}{m_1 + m_2}\right) r\dot{\theta}^2 = 0.75g$.

QUESTION 1



(a) $\underline{r} = r\underline{e}_r \Rightarrow \underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$

$T = \frac{1}{2}m\underline{v}\cdot\underline{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$\underline{H}_o = \underline{r} \times m\underline{v} = mr^2\dot{\theta}\underline{E}_z$

(b) $\dot{\underline{H}}_o = \underline{r} \times \underline{F} = \underline{0}$ as $\underline{r} \parallel \underline{F} \Rightarrow \underline{H}_o$ is constant $\Rightarrow mr^2\dot{\theta}$ is constant

(c) $\dot{T} = \underline{F}\cdot\underline{v} = -\frac{Gmm}{r^2}\dot{r} = -\frac{d}{dt}\left(-\frac{Gmm}{r}\right) \Rightarrow \frac{d}{dt}\left(T + \frac{Gmm}{r}\right) = 0$

$\Rightarrow$ Total energy $E = T + \frac{Gmm}{r}$ is conserved

(d) During the satellites motion E and h are conserved

$\Rightarrow \quad E_o = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{Gmm}{r}$ and $h = mr^2\dot{\theta}$

$\Rightarrow \quad E_o = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{h}{mr^2}\right)^2 - \frac{Gmm}{r}$

$\Rightarrow \quad \dot{r}^2 = v^2 = \frac{2E_o}{m} + \frac{2Gm}{r} - \frac{h^2}{2m^2r^2}$

QUESTION 2

(a) By definition: $\quad (m_1+m_2)\underline{r} = m_1\underline{r}_1 + m_2\underline{r}_2$

Hence $\quad (m_1+m_2)\underline{r} = m_1\underline{r} + m_2\underline{r} + m_2 r_2 \underline{e}_r + m_1 r_1 \underline{e}_r$

$\Rightarrow \quad m_2 r_2 = m_1 r_1 \quad (*)$

Now $\quad r = r_1 + r_2 \Rightarrow r = r_1 + \frac{m_1}{m_2} r_1 \Rightarrow r_1 = \left(\frac{m_2}{m_1+m_2}\right) r \quad (**)$

we also find that $\quad r_2 = \left(\frac{m_1}{m_1+m_2}\right) r$ by combining $(*)$ + $(**)$

Finally $\quad m_1 r_1^2 + m_2 r_2^2 = \left(\frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_1^2 m_2}{(m_1+m_2)^2}\right) r^2 = \left(\frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2}\right) r^2$

$$= \left(\frac{m_1 m_2}{m_1+m_2}\right) r^2 \qquad (***)$$

(b) By definition $\quad \underline{G} = m_1 \dot{\underline{v}}_1 + m_2 \underline{v}_2 = (m_1+m_2)\underline{v} = (m_1+m_2)\dot{\underline{r}}$

$\underline{H}_O = \underline{H}_c + \underline{r} \times m\underline{v}$

Now $\quad \underline{v}_1 = v - \cancel{\dot{r}_1}\underline{e}_r - r_1\dot{\theta}\underline{e}_\theta$, $\quad \dot{r}_1 = 0$

$\underline{v}_2 = \underline{v} + \cancel{\dot{r}_2}\underline{e}_r + r_2\dot{\theta}\underline{e}_\theta$, $\quad \dot{r}_2 = 0$

Now $\quad \underline{H}_c = (\underline{r}_1 - \underline{r}) \times m_1\underline{v}_1 + (\underline{r}_2 - \underline{r}) \times m_2\underline{v}_2$

$= -\underline{r}_1\underline{e}_r \times m_1(\underline{v} - \cancel{\dot{r}_1}\underline{e}_r - r_1\dot{\theta}\underline{e}_\theta)$

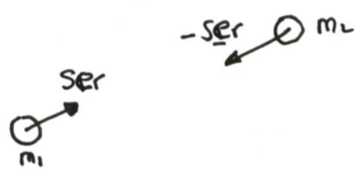
$+ r_2\underline{e}_r \times m_2(\underline{v} + \cancel{\dot{r}_2}\underline{e}_r + r_2\dot{\theta}\underline{e}_\theta)$

$= (-m_1 r_1 + m_2 r_2)\underline{e}_r \times \underline{v} + (m_1 r_1^2\dot{\theta} + m_2 r_2^2\dot{\theta})\underline{E}_z$

$= \left(\frac{m_1 m_2}{m_1+m_2}\right) r^2\dot{\theta}\,\underline{E}_z \quad$ using $(***)$ and $(*)$

Hence $\quad \underline{H}_O = \left(\frac{m_1 m_2}{m_1+m_2}\right) r^2\dot{\theta}\,\underline{E}_z + (m_1+m_2)(x\dot{y} - y\dot{x})\,\underline{E}_z$

(c)



(d) $\dot{T} = \underline{F}_1 \cdot \underline{v}_1 + \underline{F}_2 \cdot \underline{v}_2 = S\underline{e}_r \cdot (\underline{v}_1 - \underline{v}_2)$

$= S\underline{e}_r \cdot (-r_1\dot{\theta}\underline{e}_\theta + r_2\dot{\theta}\underline{e}_\theta) = 0$

$\Rightarrow$ T is conserved

$\dot{\underline{G}} = \underline{F}_1 + \underline{F}_2 = S\underline{e}_r - S\underline{e}_r = \underline{0}$

$\Rightarrow$ $\underline{G}$ is conserved

(e) $\dot{\underline{H}}_c = -r_1\underline{e}_r \times S\underline{e}_r + r_2\underline{e}_r \times -S\underline{e}_r = \underline{0} \Rightarrow \underline{H}_c$ is constant $\Rightarrow \frac{m_1 m_2}{m_1+m_2} r^2\dot{\theta}$ is constant

(f) We wish for $\|\dot{\underline{v}}_1\| = g(3/4)$. Now $\dot{\underline{v}}_1 = \dot{\underline{v}} + r_1\dot{\theta}^2\underline{e}_r = r_1\dot{\theta}^2\underline{e}_r$ $\quad$ Hence, we want $r_1\dot{\theta}^2 = g\left(\frac{3}{4}\right)$

But $r_1 = \frac{m_2}{m_1+m_2} r$. Hence $\left(\frac{m_2}{m_1+m_2}\right) r\dot{\theta}^2 = g\left(\frac{3}{4}\right)$.