First Midterm Examination
Closed Books and Closed Notes

Question 1
A Planar Pendulum (25 POINTS)

As shown in Figure 1, a particle of mass m is attached to a fixed point O by an inextensible string of length L. The motion of the particle is in the $\mathbf{E}_x - \mathbf{E}_y$ plane.

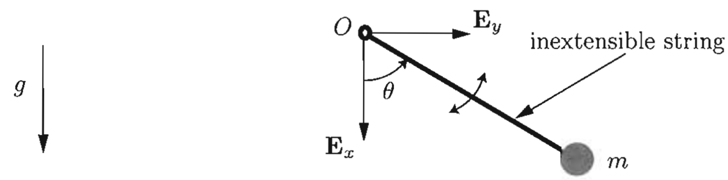


Figure 1: *Schematic of a particle of mass m which is attached to a fixed point O by an inextensible string of length L. A vertical gravitational force $mg\mathbf{E}_x$ acts on the particle.*

(a) Starting from the usual representation for the position vector $\mathbf{r} = L\mathbf{e}_r$, establish expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. For the two cases where $\dot{\theta} < 0$ and $\dot{\theta} > 0$, what are the unit tangent $\mathbf{e}_t$ and unit normal $\mathbf{e}_n$ vectors to the path of the particle? Illustrate your answers with a sketch.

(b) Draw a freebody diagram of the particle.

(c) Show that the tension force $\mathbf{T}$ acting on the particle is

$$\mathbf{T} = -mL\left(\frac{g}{L}\cos(\theta) + \dot{\theta}^2\right)\mathbf{e}_r, \tag{1}$$

In addition, show that the equation of motion of the particle is

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta). \tag{2}$$

(d) Suppose the particle is given an initial speed $L\dot{\theta}_0$ when $\theta = \theta_0 = \frac{\pi}{2}$. Show for the ensuing motion that

$$\dot{\theta}^2(\theta) = \dot{\theta}_0^2 + \frac{2g}{L}\cos(\theta). \tag{3}$$

(e) What is the minimum value of $\dot{\theta}_0$ required so that the string will not become slack during the ensuing motion?

Question 2

A Particle on a Cosinusoidal Track (25 POINTS)

As shown in Figure 2, a bead of mass m moves on a thin circular rod that is rough. The equation for the centerline of the rod is given by the equation $y = \alpha\cos(x)$ where α is a constant. The bead is connected to a fixed point A by a linear spring of stiffness K and unstretched length L_0. The contact between the bead and the rod is rough with a coefficient of static friction μ_s and a coefficient of kinetic friction of μ_k. In addition to friction, spring, and normal forces, a vertical gravitational force $-mg\mathbf{E}_y$ acts on the bead.

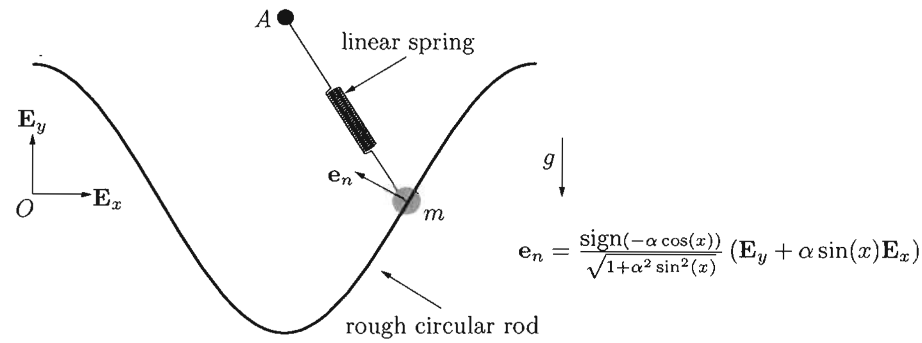


Figure 2: *Schematic of a particle of mass m moving on a rough guide.*

(a) Using a Cartesian coordinate system, the position vector of the particle is

$$\mathbf{r} = x\mathbf{E}_x + \alpha\cos(x)\mathbf{E}_y. \tag{4}$$

Derive expressions for the speed v, and velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. What is the unit tangent vector $\mathbf{e}_t$ to the curve that the bead is moving on?

(b) Draw a freebody diagram of the particle. Give clear expressions for the forces acting on the particle, and distinguish the static friction and dynamic friction cases.

(c) Suppose that the particle is moving on the curve with $\dot{x} > 0$. Show that the equation governing the motion of the particle is

$$m\dot{v} = -\mu_k \|\mathbf{N}\| + \frac{mg\alpha\sin(x)}{\sqrt{1+\alpha^2\sin^2(x)}} + \mathbf{F}_s \cdot \mathbf{e}_t, \tag{5}$$

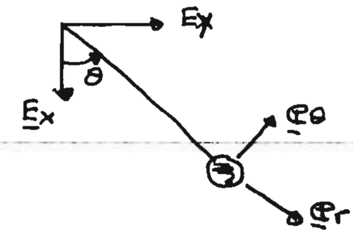
where $\mathbf{F}_s$ is the spring force and $\mathbf{N}$ is the normal force. How would you solve for $\mathbf{N}$?

(d) Suppose that the particle is stationary at a point $x = x_0$ on the curve. In the absence of a spring force, show that this implies that

$$|\alpha\sin(x_0)| \leq \mu_s. \tag{6}$$

If $\mu_s = \frac{1}{\sqrt{2}}$ and $\alpha = 1$, then illustrate the possible locations x_0.

QUESTION 1



(a) $\underline{r} = L\,\underline{e}_r$

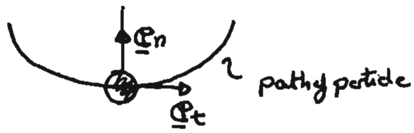
$\underline{v} = L\dot{\theta}\,\underline{e}_\theta \qquad \text{as} \quad \dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta$

$\underline{a} = L\ddot{\theta}\,\underline{e}_\theta - L\dot{\theta}^2\,\underline{e}_r \qquad \text{as} \quad \dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r$

$\underline{e}_r = \cos\theta\,\underline{E}_x + \sin\theta\,\underline{E}_y$

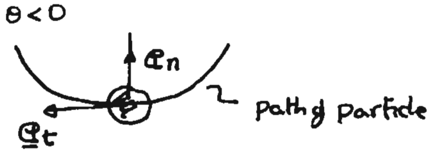
$\underline{e}_\theta = -\sin\theta\,\underline{E}_x + \cos\theta\,\underline{E}_y$

$\dot{\theta} > 0$



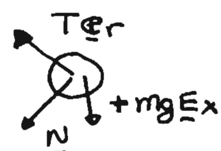
$\underline{e}_t = \underline{e}_\theta, \quad \underline{e}_n = -\underline{e}_r$

$\dot{\theta} < 0$



$\underline{e}_t = -\underline{e}_\theta, \quad \underline{e}_n = -\underline{e}_r$

(b)



$mg\,\underline{E}_x = mg\,(\cos\theta\,\underline{e}_r - \sin\theta\,\underline{e}_\theta)$

$\underline{N} = N\,\underline{E}_z$

(c) $\underline{F} = m\underline{a}$

$\cdot\,\underline{e}_r \qquad T + mg\cos\theta = -mL\dot{\theta}^2 \quad \Rightarrow \quad \underline{T} = T\underline{e}_r = -mL\left(\frac{g}{L}\cos\theta + \dot{\theta}^2\right)\underline{e}_r$

$\cdot\,\underline{e}_\theta \qquad -mg\sin\theta = mL\ddot{\theta} \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{L}\sin\theta$

$\cdot\,\underline{E}_z \qquad N = 0.$

(d) From $\ddot{\theta} = -\frac{g}{L}\sin\theta$ we use the identity $\ddot{\theta}(\theta) = \frac{d\dot{\theta}}{d\theta}\dot{\theta}$

$\Rightarrow \int_{\theta_0}^{\theta} -\frac{g}{L}\sin u\,du = \int_{\dot{\theta}_0}^{\dot{\theta}} x\,dx \quad \Rightarrow \quad \frac{1}{2}\dot{\theta}^2 = \frac{1}{2}\dot{\theta}_0^2 + \left(\frac{g}{L}\cos u\right)\Big|_{\theta_0 = \pi/2}^{\theta}$

$\Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_0^2 + \frac{2g}{L}\cos\theta.$

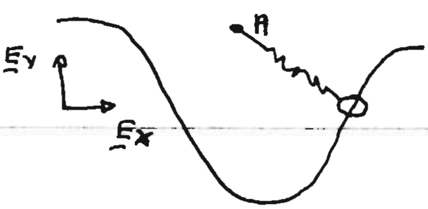
(e) Using results from (c) & (d) $\underline{T} = -mL\left(\dot{\theta}_0^2 + \frac{3g}{L}\cos\theta\right)\underline{e}_r$

String becomes slack when $\dot{\theta}_0^2 + \frac{3g}{L}\cos\theta \leq 0$

This happens when $\cos\theta \leq -\frac{L}{3g}\dot{\theta}_0^2$

So if $\dot{\theta}_0^2 \geq \frac{3g}{L}$, then the string can never become slack. even when the string is vertical $(\theta = \pi)$

QUESTION 2



(a)

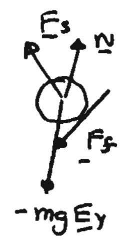
$\underset{\sim}{r} = x \underline{E}_x + \alpha \cos x \, \underline{E}_y$

$\underline{v} = \dot{x}(\underline{E}_x - \alpha \sin x \, \underline{E}_y) \quad \Rightarrow \quad v = |\dot{x}| \sqrt{1+\alpha^2 \sin^2 x}$

$\underline{e}_t = \dfrac{\underline{v}}{v} = \dfrac{\dot{x}}{|\dot{x}|} \dfrac{1}{\sqrt{1+\alpha^2\sin^2 x}} \left(\underline{E}_x - \alpha \sin x \, \underline{E}_y \right)$

$\underline{a} = \ddot{x}(\underline{E}_x - \alpha \sin x \, \underline{E}_y) - \dot{x}^2 \alpha \cos x \, \underline{E}_y$

(b)



$\underline{F}_s = -K\left(\|\underline{r}-\underline{r}_A\| - L\right) \dfrac{\underline{r}-\underline{r}_A}{\|\underline{r}-\underline{r}_A\|}$

$\underline{F}_f = F_f \underline{e}_t$ ... static case

$= -\mu_K \|\underline{N}\| \dfrac{\underline{v}}{\|\underline{v}\|} = -\mu_K \|\underline{N}\| \underline{e}_t$ ... dynamic case

$\underline{N} = N_n \underline{e}_n + N_b \underline{e}_b$

(c)

$\underline{F} = m\underline{a}$

$\cdot \underline{e}_t \qquad m\dot{v} = -\mu_K \|\underline{N}\| - mg \underline{E}_y \cdot \underline{e}_t + \underline{F}_s \cdot \underline{e}_t$

$= -\mu_K \|\underline{N}\| + \dfrac{mg\,\alpha \sin x}{\sqrt{1+\alpha^2 \sin^2 x}} + \underline{F}_s \cdot \underline{e}_t \qquad$ where $\dot{x} > 0$

$\cdot \underline{e}_n \qquad m v^2 \kappa = N_n - mg \underline{E}_y \cdot \underline{e}_n + \underline{F}_s \cdot \underline{e}_n$

$\cdot \underline{e}_b \qquad N_b = 0$

Hence $\quad \underset{\sim}{N} = N \underline{e}_n$ and $N = -\underline{F}_s \cdot \underline{e}_n + \dfrac{mv^2}{\rho} + \dfrac{mg\,\mathrm{sign}(-\alpha \cos x)}{\sqrt{1+\alpha^2 \sin^2 \alpha x}}$

(d) Stationary $\Rightarrow$ Static friction. $\Rightarrow \quad F_f = -\dfrac{mg\,\alpha \sin x_0}{\sqrt{1+\alpha^2 \sin^2 x_0}}$ and $N = mg \underline{E}_y \cdot \underline{e}_n$

But $\|\underline{F}_f\| \le \mu_s \|\underline{N}\| \quad \Rightarrow \quad |-mg\alpha \sin x_0| \le \mu_s \,|mg\,\mathrm{Sign}(\alpha \cos x_0)|$

$\Rightarrow \quad |\alpha \sin x_0| \le \mu_s$

$\mu_s = \dfrac{1}{\sqrt{2}}$

$\alpha = 1$