Math 104: Introduction to Analysis

Final exam, May 24th, 2002

Weingart

Name: ______________________________

Signature: ___________________________

There are 10 problems on this final worth 20 points each, however you should not work on more than 9 problems of your choice dropping the last one. In any case you will only get credit for 9 of the 10 problems. Show as much of your work as possible to receive credit. Successful final!

1	2	3	4	5	6	7	8	9	10	Total

Codename: ______________________________

Problem 1: (20 points)
Find the Taylor series of $f(x) := 5x^3 + 6x^2 - 7x + 2$ in $x_0 = 1$ and its radius of convergence.

Problem 2: (20 points)
Consider a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ which is differentiable on (a, b) with bounded derivative $|f'(x)| \leq C, x \in (a, b)$. Show that f is Lipschitz continuous on $[a, b]$ with Lipschitz constant C.

Problem 3: (20 points)
Give precise formulations for the Theorem of Heine–Borel, the Mean Value Theorem, the Intermediate Value Theorem and one version of the Fundamental Theorem of Calculus.

Problem 4: (20 points)
Monotone increasing functions $F : [a, b] \longrightarrow \mathbb{R}$ serve as integrators in the theory of Riemann–Stieltjes integrals. Recall that we defined

$$dF([c,d]) \ := \ \lim_{x \longrightarrow d^+} F(x) \ - \ \lim_{x \longrightarrow c^-} F(x) \, .$$

Show that the sum of two monotone increasing functions F and G is again monotone increasing with $d(F+G) = dF + dG$.

Problem 5: (20 points)
Show that

$$\lim_{n \longrightarrow \infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \ldots + \frac{n}{n^2 + n^2} \right) = \frac{\pi}{4}.$$

Problem 6: (20 points)
Consider the sequence of functions $(f_n)_{n \geq 1}$ on $[0, 1]$ defined by writing $n = 2^k + m$ with $k \geq 0$ and $0 \leq m < 2^k$ and setting

$$f_n(x) := \begin{cases} 1 & \text{for } x \in [\frac{m}{2^k}, \frac{m+1}{2^k}] \\ 0 & \text{else} \end{cases}$$

Draw the graphs of a few f_n to see what is going on and show that

$$\lim_{n \longrightarrow \infty} \int_0^1 f_n^2 = \lim_{n \longrightarrow \infty} \int_0^1 (f_n - 0)^2 = 0$$

although f_n does not converge pointwise to the zero function (hence not uniformly either).

Problem 7: (20 points)
Let f be a (Darboux) integrable function on $[a,b]$ and F a differentiable function on $[a,b]$ with $F'(x) = f(x)$ except for finitely many $x \in [a,b]$. Show that F' is integrable as well and conclude:

$$\int_a^b f \;=\; F(b) - F(a)\,.$$

Problem 8: (20 points)
Consider a uniformly continuous function $g:\ \mathbb{R} \longrightarrow \mathbb{R}$ and a sequence of functions $(f_n)_{n\geq 1}$ on $[a,b]$ converging uniformly to a function $f:\ [a,b] \longrightarrow \mathbb{R}$. Prove that the sequence $(g \circ f_n)_{n\geq 1}$ of functions converges uniformly to the composition $g \circ f$ with $(g \circ f)(x) := g(f(x))$.

Problem 9: (20 points)
Let $\gamma : [0,1] \longrightarrow \mathbb{R}^3$ be a continuous path with $\gamma(0) = (0,0,0) \in \mathbb{R}^3$ and $\gamma(1) = (1,1,1)$. Show that γ meets the unit sphere $S := \{ (x,y,z) \in \mathbb{R}^3 | \, x^2 + y^2 + z^2 = 1 \}$ in $\mathbb{R}^3$.

Problem 10: (20 points)
Give a complete proof of the integral criterion for convergence of series. Namely for a monotone decreasing function $f : [0,\infty) \longrightarrow \mathbb{R}$ with $f(x) \geq 0$ for all $x \geq 0$ and

$$\lim_{b \longrightarrow \infty} \int_0^b f \; < \; \infty$$

the series $\sum_{m=1}^{\infty} f(m)$ converges (absolutely).