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Fall 2000, Math 104, Section 2

First Midterm

3 October, 2000
11:10-12:30 PM

1. (28 points, 7 points apiece) Complete each of the following definitions. (Do not give examples or other additional facts about the concepts defined.)

(a) A *metric space* is

(b) A set X is said to be *countable* if

(c) The *radius of convergence* R of the power series $\Sigma_{n=0}^{\infty} c_n z^n$ is defined to be

(d) If S is an ordered set, E a subset of S, and x an element of S, then we call x the *least upper bound* of E, and write $x = \sup E$, if

2. (40 points; 10 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated.)

(a) A non-convergent Cauchy sequence (a_n) in a metric space X.

(b) A sequence of points in the interval $[0,1] \subseteq R$ having no convergent subsequence.

(c) A convergent series $\Sigma\, a_n$ such that $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$.

(d) A family of subsets $G_\alpha \subseteq R$ (where α ranges over some set A) such that every finite subfamily $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ has nonempty intersection, but the whole family has empty intersection, $\bigcap_{\alpha\in A} G_\alpha = \varnothing$.

3. (32 points) Let X be a metric space and K a compact subset of X.

(a) (12 points) Show that for every real number $\varepsilon > 0$ there exists a finite subset $S \subseteq K$ such that $(\forall x \in K)(\exists s \in S)\ d(x, s) < \varepsilon$; i.e., such that each point of K is within distance $< \varepsilon$ of some point of S.

(b) (20 points) Deduce from the result of (a) that K contains a finite or countable subset T which is dense in K. (Recall that a subset of K is called *dense* if every point of K is a member of the subset or a limit point of the subset. Suggestion: apply part (a) to a countable sequence of values of ε which approach 0.) In doing this part you may assume the result of (a) even if you did not succeed in proving it.