

1. (30 points, 4 points each except for (d).) Complete each of the following definitions. (Do not give examples or other additional facts about the concepts defined.)

(a) If (p_n) is a sequence of points of a metric space X and q a point of X , then we say that the sequence (p_n) converges to q (or that $\lim_{n \rightarrow \infty} p_n = q$), if

(b) If (a_n) is a sequence of real numbers, and s is a real number, then we say that $\sum_{n=1}^{\infty} a_n = s$ (or that the series $\sum_{n=1}^{\infty} a_n$ converges to s) if

(c) If X and Y are metric spaces, a function $f: X \rightarrow Y$ is said to be *uniformly continuous* if

(d) (2 points) Let γ be a curve in R^k . Assume the definition of its length $\Lambda(\gamma)$ known. Then γ is said to be *rectifiable* if

(e) Let E be a set, (f_n) a sequence of complex-valued functions on E , and f a complex-valued function on E . Then (f_n) is said to *converge uniformly* to f if

(f) If f is a continuous bounded complex-valued function on a metric space X , then $\|f\|$ is defined to be

(g) If \mathcal{A} is a set of complex-valued functions on a set E , then the *uniform closure* of \mathcal{A} means

(h) A set \mathcal{A} of complex-valued functions on a metric space X is said to be *equicontinuous* if

2. (30 points; 5 points each.) For each of the items listed below, either *give an example*, or give a brief reason why *no example exists*. (If you give an example, you do *not* have to prove that it has the property stated.)

(a) A subset of R that is neither open nor closed.

(b) A sequence (a_n) of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} (1-a_n)$ both converge.

(c) A differentiable function $f: R \rightarrow R^2$ such that $f(0) = f(1)$, but such that $f'(x)$ is nonzero for all $x \in [0, 1]$.

(d) A real-valued Riemann-integrable function f on $[0, 1]$ such that the function $\sin(f(x))$ is not Riemann-integrable.

(e) A sequence (f_n) of uniformly continuous functions on a metric space X which converges pointwise to a function which is discontinuous.

(f) A discontinuous real-valued function on $[-1, 1]$ which can be written as the limit of a uniformly convergent sequence of polynomial functions.

3. (10 points) Suppose K and L are compact subsets of a metric space X . Show that $K \cup L$ (the union of K and L) is also compact.

(If you give a proof that only applies to subsets of Euclidean spaces R^k , you will get partial credit if you say explicitly that you are only covering this case, and if your argument is detailed and correct.)

4. (30 points) I give below a theorem from Rudin and its proof (both slightly reworded), and will ask you to justify certain steps of the proof. Namely, wherever you encounter a Roman numeral in a box, e.g., iii, before an assertion in the proof, answer the question about that assertion posed after the corresponding symbol in the right-hand column. (On this compact version of the exam, the questions are given in parentheses, generally after the sentence containing the box.)

If the answer to some part is continued on the back of the page, you can signal this by writing \rightarrow . If you cannot justify some step, you may nonetheless assume it in your justifications of later steps. I recommend reading the whole proof through before starting on the questions.

Theorem Suppose (f_n) is a sequence of real-valued differentiable functions on $[a, b]$ such that the sequence of derivatives (f'_n) converges uniformly on $[a, b]$, and such that for some $x_0 \in [a, b]$, the sequence of real numbers $(f_n(x_0))$ converges. Then (f_n) converges uniformly on $[a, b]$ to a differentiable function f , such that for all $x \in [a, b]$,

$$(1) \quad f'(x) = \lim_{n \rightarrow \infty} f'_n(x).$$

Proof Let $\varepsilon > 0$ be given. Then [i] there exists N_1 such that whenever n and m are both $\geq N_1$

$$(2) \quad |f_n(x_0) - f_m(x_0)| < \varepsilon/2,$$

([i] What assumption of the theorem implies that such N_1 exists?) and [ii] there exists N_2 such that whenever n and m are both $\geq N_2$ and $t \in [a, b]$,

$$(3) \quad |f'_n(t) - f'_m(t)| < \varepsilon/(2(b-a)).$$

([ii] What assumption of the theorem implies that such N_2 exists?) Thus [iii] there exists N such that for all n and $m \geq N$, both (2) and (3) hold. ([iii] How can such N be chosen?)

[iv] The Mean Value Theorem, applied to the function $f_m - f_n$, implies that for distinct x and t in $[a, b]$, and integers n and m both $\geq N$, ([iv] Write the equation gotten by applying the Mean Value Theorem to $f_m - f_n$ for endpoints x and t .)

$$(4) \quad |f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq |x-t| \varepsilon/(2(b-a)) \leq \varepsilon/2.$$

([v] How do we get this first inequality of (4) from the equation of [iv]?) ([vi] Why does this second inequality of (4) hold?) We also note that for any x and t in $[a, b]$, and any integers n and m , the triangle inequality gives

$$(5) \quad |f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(t) + f_m(t)| + |f_n(t) - f_m(t)|.$$

([vii] To what A and B is the triangle inequality, $|A+B| \leq |A|+|B|$, being applied?) Hence setting $t = x_0$ in (5) and applying (4) and (2) to the right side, we see that for n and $m \geq N$ and $x \in [a, b]$,

$$(6) \quad |f_n(x) - f_m(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that [viii] (f_n) is a Cauchy sequence, ([viii] With respect to what distance-function?) hence it converges uniformly to some function

$$(7) \quad f = \lim_{n \rightarrow \infty} f_n.$$

Let us now fix a point $x \in [a, b]$ and define, for all $t \neq x$ in $[a, b]$,

$$(8) \quad g_n(t) = (f_n(t) - f_n(x))/(t-x),$$

and

$$(9) \quad g(t) = (f(t) - f(x))/(t-x).$$

Then the first inequality of (4) shows that

$$(10) \quad |g_n(t) - g_m(t)| \leq \varepsilon/(2(b-a)),$$

([ix] Say briefly how one obtains (10) from the first inequality of (4).) which says that the sequence of functions (g_n) converges uniformly on the set where it is defined, namely the set of points $t \in [a, b]$ other than x . Since the g_n converge pointwise to g by (8), (9) and (7), the function to which they converge uniformly must also be g .

We see from (8) that for all positive integers n ,

$$(11) \quad \lim_{t \rightarrow x} g_n(t) = f'_n(x)$$

([x] Why?) and similarly from (9) that

$$(12) \quad \lim_{t \rightarrow x} g(t) = f'(x) \text{ if and only if either side exists.}$$

Now recall that Theorem 7.11 says that if a sequence of functions (g_n) on a subset E of a metric space X converges uniformly to a function g on E , and if each function g_n approaches a limit A_n as the variable approaches some $x \in X$, then g also approaches a limit A as the variable approaches x , and

$$(13) \quad A = \lim_{n \rightarrow \infty} A_n.$$

([xi] Applying that theorem here, ([xi] Which preceding statement showed that our functions approached limits A_n , and what are those limits?) we conclude that the left-hand side of (12) exists, and equals $\lim_{n \rightarrow \infty} f'_n(x)$, giving the desired equation (1).