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Math 104-2: Midterm 1

NOTE: In retrospect, this exam was much too long!

This exam has two sections:

- Section I is short, and all the problems have short answers. You don't need to justify or prove anything – just give your solution and move on. The entire section is worth 30 points, and I don't recommend spending more than 10 or 15 minutes on it.
- Section II consists of longer problems, in which proofs are essential. You need to do one problem from IIA and one from IIB. Make sure to indicate clearly which of the problems you are submitting solutions for. Otherwise, if you worked on more than one problem from either IIA or IIB, I will choose at random which one to look at; you're much better off making that decision for yourself!

You may use all the basic arithmetic manipulations without proof and without mention. You may also use any result proved in class, in the book, or in a homework problem – but these must be *mentioned explicitly.* As a general rule, your job is to convince me that every time you choose not to explain something in excruciating detail, it is the result of knowledge and understanding on your part, not ignorance or uncertainty.

I will give a lot of partial credit to partial arguments; however, mere lists of facts (even true facts) that are going nowhere will receive very little, if any, credit. I may also take off points for a valid argument that is extremely poorly organized. (Don't worry, I know you're under time pressure – I won't be too picky!) Arguments based on intuitive concepts or pictures will certainly get you partial credit, but for full credit, a complete formal proof required.

Time limit on exam: 50 minutes

Good Luck!

Section I

1. *(5 pts each)* True or False?

   (a) $\mathbb{Z}$ has the least-upper-bound property.

   (b) Compact sets are never open.

2. *(5 pts)* What are the limit points in $\mathbb{R}$ of the set $\{x : x > 17\}$?

3. *(5 pts)* Suppose I try to define a metric on $\mathbb{R}^k$ by

$$d(p, q) = |p| + |q|.$$

   Which of the axioms of a metric space does this function *fail* to satisfy?

4. *(5 pts each)* Give an example. **CHOOSE TWO OUT OF THREE.**

   (a) A compact, countable subset of $\mathbb{R}^2$.

   (b) A subset of $\mathbb{Q}$ that is neither open nor closed.

   (c) A compact, nonempty subset of $\mathbb{Q}$.

Section II

A. Do **ONE** of the following:

1. Let A and B be subsets of $\mathbb{R}$ that are nonempty and bounded above. Define

$$A + B = \{x \in \mathbb{R} : x = a + b \text{ for some } a \in A,\ b \in B\}.$$

Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

2. The following statement should sound familiar (it's logic problem #26):

> *If P and Q are any two nonempty sets of numbers, and if for every member x of P and every member y of Q, we have $x < y$, then it is possible to find a number w such that whenever x is a member of P and y is a member of Q, we have $x \leq w \leq y$.*

Show that this statement is true if by "numbers" we mean "elements of $\mathbb{R}$", and false if we mean "elements of $\mathbb{Q}$".

Section II

B. Do **ONE** of the following (see this page and the next):

1. Consider the following statement about a metric space X and an arbitrary point $p \in X$:

   (*) *For any* $r > 0$, $\overline{N_r(p)} = \{q \in X : d(p, q) \leq r\}$.

   (a) Prove that (*) is true when $X = \mathbb{R}^k$ and $p = 0$. *(Hint: remember that points in* $\mathbb{R}^k$ *can be added and multiplied by numbers. If you multiply a point* $q \in \mathbb{R}^k$ *by a number slighly less than* 1*, you obtain a point slightly closer to* 0*.)*

   In case you were wondering, there is actually nothing special about $p = 0$: when $X = \mathbb{R}^k$, (*) is true for any p. It's just a little messier, but fundamentally no harder to show.

   (b) Is (*) true when $X = \mathbb{Q}$?

   (c) Is (*) true when X is a finite set?

2. **(Note: This problem looks long, but that's only because it consists almost entirely of hints! Without these hints it would be hard, but as it stands, I don't think it's any harder than Problem B1. At any stage in the proof, feel free to use the statements in the previous parts, even if you don't know how to prove them.)**

In a metric space X, it is often useful to talk about distance not just between points, but between a point and a set. Intuitively, we want this to be the smallest distance between the point and anything in the set; so we make the following

Definition: If $p \in X$ and $E \subset X$, define $d(p, E) = \inf\{d(p,q) : q \in E\}$.

(a) Show that for any X, p, and E as above, $d(p, E)$ makes sense (i.e. exists) as long as $E \neq \emptyset$.

The rest of this problem walks you through the proof of the following

Theorem: If $E \neq \emptyset$ and E is compact, then for any $p \in X$ there exists a point $q \in E$ such that $d(p, E) = d(p, q)$.

Proof: Suppose no such point q exists.

(b) Let $D = d(p, E)$. Show that for every $n \in \mathbb{N}$, we can find a point $q_n \in E$ such that $d(p, q_n) < D + \frac{1}{n}$.

(c) Although it is possible that we picked $q_n = q_m$ for $n \neq m$, show that the set $Q = \{q_1, q_2, q_3, \ldots\}$ is nonetheless infinite. (Hint: assume it's finite and derive a contradiction.)

(d) Suppose x is any point of E. By assumption, $d(p, x) > D$, so $d(p, x) = D + \epsilon$ for some $\epsilon > 0$. Show that if $n > \frac{2}{\epsilon}$ then $d(x, q_n) > \frac{\epsilon}{2}$.

(c) Use (d) to show that x cannot be a limit point of Q. (Hint: the number

$$\min\left(\frac{\epsilon}{2},\ d(x, q_1),\ d(x, q_2), \ldots,\ d(x, q_N)\right)$$

should figure somewhere in your answer. You just have to decide how to use it, and what N should be.)

(f) Derive a contradiction from the fact that E is compact! (This is the only place in the proof where we use the compactness of E.) **QED**