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Fall 2000, Math 104, Section 2 — 7 November, 2000

Second Midterm — 11:10-12:30 PM

1. (20 points, 10 points apiece) Complete each of the following definitions. (Do not give examples or other additional facts about the concepts defined. Note that in each definition there is both a short phrase and a longer part to be filled in.)

(a) If X is a metric space and p a point of X, then a function f from X to .... is said to have a *local maximum* at p if

(b) If α is .... on an interval $[a, b]$, then $\mathscr{R}(\alpha)$ denotes the set of all real-valued functions f which

2. (40 points; 8 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated.)

(a) A function f with a discontinuity of the second kind.

(b) A real-valued function f on $[0, 1]$ (not necessarily continuous) which does not attain a maximum value.

(c) A function $f: R \to R$ such that as $x \to +\infty$, $f(x)$ does not approach a real number, nor $+\infty$, nor $-\infty$.

(d) A differentiable function $\mathbf{f}: R \to R^2$ such that $\mathbf{f}(0) = \mathbf{f}(1)$, but such that $\mathbf{f}'(x)$ is nonzero for all $x \in [0, 1]$.

(e) A monotonically decreasing real-valued function f on $[0,1]$ which is not Riemann integrable.

3. (20 points) Let $f: R \to R$ be differentiable. Show that if f' is bounded (i.e., if there exists a real number M such that $|f'(x)| \leq M$ for all $x \in R$), then f is uniformly continuous. (Recall that this conclusion means that $(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x, y)\ (|x - y| < \delta \Rightarrow |f(x) - f(x)| < \varepsilon)$.)

4. (20 points) Let α be an increasing function, and f_1, f_2 arbitrary functions, on an interval $[a, b]$. Show that

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f_1 + f_2, \alpha).$$

(Rudin writes this inequality without explanation in his proof that the sum of integrable functions is integrable. Suggestion: Use the definitions of these lower sums; in particular, of the terms "m_i" they involve. You might call the terms in the three lower sums $m_i^{(1)}$, $m_i^{(2)}$ and m_i^{+} respectively; or simply replace them by their definitions in your proof.)