Math 104, Section 3 (Curtin). **FINAL EXAM**

I recommend that you look over the problems and do those which you can first–there is no particular logic in their ordering. However, be sure to clearly label the problems in your blue books so that I can find them.

Be sure to include explanations which justify each step in your arguments and computations. For example, check that the hypotheses of any theorem that you use are satisfied. Do not interpret the problems in such a way that they become trivial–if in doubt, ask.

Should it become necessary to leave the room during this exam (eg. fire alarm), this exam and all your work is to remain in the room, face down on your desk.

Throughout this exam **R** denotes the real numbers.

1. Let $f : [-1,1] \to \mathbf{R}$ be defined by
$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some integer } n, \\ 0 & \text{otherwise.} \end{cases}$$
Compute or show that the following limits do not exist: $\lim_{x\to 1/3} f(x)$ and $\lim_{x\to 0} f(x)$.

2. Let $f(x) = x|x|$ for all real numbers x. Compute $f'(x)$ and $f''(x)$ and state the domains where they exist.

3. Show that if $\sum a_k$ converges absolutely, then $\sum a_k^2$ converges.

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and that $f'(x)$ exists and is bounded on $\mathbf{R}$. Show that f is uniformly continuous on $\mathbf{R}$.

5. Prove that if $\sum g_k$ converges uniformly on a set S and if h is a bounded function of S, then $\sum hg_k$ converges uniformly on S.

6. Find the Taylor series for $f(x) = |x-1|$ about zero. What is its radius of convergence? Where does it agree with f? Is there any power series which agrees with $f(x)$ for all $x \in \mathbf{R}$? Why or why not?

7. Let $g(x) = x^2$ for rational x and $g(x) = 0$ for irrational x. Calculate the upper and lower Darboux integrals for f on the interval $[0,1]$. Is g integrable?

8. Suppose that f is differentiable on $\mathbf{R}$, that $1 \le f'(x) \le 2$ for all $x \in \mathbf{R}$, and that $f(0) = 0$. Prove that $x \le f(x) \le 2x$ for all $x \ge 0$.

9. Suppose that f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in [a,b]$ such that $f(x) = g(x)$.

10. Prove that if (a_n) is a bounded nondecreasing sequence then it converges. (This is a special case of a theorem in the book, and you will get no credit for only making this observation).