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MATH 104-2: Final Exam

Section I. Do 5 problems (9 points each).

1. Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.
2. Let α be a bounded, monotonically increasing function on $\mathbb{R}$. What is

$$\int_{-\infty}^{\infty} 1 \, d\alpha?$$

   (Answer and proof, please!)
3. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, define

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = n \\ 0 & \text{otherwise} \end{cases}$$

   Does $\sum f_n(x)$ converge uniformly on $\mathbb{R}$?
4. Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous and one-to-one, then it is either strictly increasing or strictly decreasing.
5. As usual, let $\mathcal{B}(\mathbb{R})$ be the set of bounded functions on $\mathbb{R}$, with metric space structure given by the supnorm $\|\cdot\|$. Define a function ϕ on $\mathcal{B}(\mathbb{R})$ by

$$\phi(f) = f(0) - f(1).$$

   Is ϕ continuous on $\mathcal{B}(\mathbb{R})$?
6. Suppose we are given the following information about a function $f : \mathbb{R} \to \mathbb{R}$:
   - f is continuous on $[0, \infty)$ and $f(0) = 0$;
   - f is differentiable on $(0, \infty)$ and f' is monotonically increasing.

   Prove that the function

$$g(x) = \frac{f(x)}{x}$$

   is monotonically increasing on $(0, \infty)$. [Hint: Differentiate!]

Section II. Do 2 problems (15 points each).

7. Suppose f is continuous on $[a, b]$ and α is continuous and *strictly* increasing.
   (a) Show that if
   $$\int_a^b f^2(x)\, d\alpha = 0,$$
   then f is identically 0 on $[a, b]$.
   (b) Show that if for every nonnegative integer n,
   $$\int_a^b f(x)x^n\, d\alpha = 0,$$
   then f is identically 0 on $[a, b]$. [Hint: use (a) and the Weierstrass Approximation Theorem.]
   (c) Give a counterexample to (b) when α is not assumed to be strictly increasing.

8. Let E be a nonempty subset of a metric space X. We can define the distance from E to a point $x \in X$ by
   $$d_E(x) = \inf\{d(z, x) : z \in E\}.$$
   Using this, we can define the distance between any two nonempty sets $E, F \subset X$:
   $$D(E, F) = \inf\{d_E(x) : x \in F\}.$$
   (a) If E is closed, show that $d_E(x) = 0$ if and only if $x \in E$.
   (b) Show that for a fixed set E, $d_E(x)$ is a continuous function on X.
   (c) Show that if E and F are compact then $D(E, F) = 0$ if and only if $E \cap F \neq \emptyset$. Give a counterexample when E and F are not assumed to be compact. [Hint: for the proof, you will need to use both (a) and (b).]

9. Prove that a power series $\sum a_n x^n$ can be differentiated term by term everywhere within its radius of convergence R. In other words, prove that on $(-R, R)$, the function
   $$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
   is differentiable, with derivative
   $$\sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Section III. Required (25 points).

10. This problem investigates the connections of the natural logarithm with the harmonic series, both regular and alternating. You will need to assume that the derivative of e^x is e^x.

    (A word on notation: mathematicians usually write "$\log x$", not "$\ln x$", for the natural log. However, since many of you probably aren't accustomed to this, I used $\ln x$ in this problem. Feel free to use either notation in your solutions.)

    (a) Prove that for all $x > 0$, the derivative of $\ln(x)$ is $\frac{1}{x}$.

    (b) Prove that
    $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$
    [Hint: Use Taylor's Theorem around 1; make sure you explain how to deal with the error term!]

    (c) Prove that the sequence whose n^{th} term is
    $$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)$$
    converges as $n \to \infty$. [Hint: write $ln(n)$ as an integral.]

    Note: The limit of this sequence is called *Euler's constant* and is usually denoted by γ. Its value is approximately .577..., but no one knows whether γ is rational or not!

    (d) Suppose it takes the flea N jumps to catch up to the kangaroo; in other words, N is the smallest integer such that
    $$\sum_{k=1}^{N} \frac{1}{k} \geq 100.$$
    How many digits (exactly!) does N have? A few people in our class tried to answer this question on their computers, and gave up after hours of useless computation. Now, however, you should be able to do this without even a calculator! All you need to know is that
    $$2.3 < \ln(10) < 2.31.$$
    Make sure you explain why your answer is precisely correct, not even off by 1!

Section IV. Extra Credit

How could we conclude a course in analysis without having proved that the derivative of e^x is e^x? Here is a sketch of a proof for you to fill in.

Consider the function

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In class we proved that this series has radius of convergence ∞. Thus by Problem 9, $\exp(x)$ is everywhere differentiable. Differentiating term by term, we find that

$$\exp'(x) = \exp(x).$$

Thus all that is left is to prove that $e^x = \exp(x)$ for all x. (Note: in PS#10, you were asked to show that $e^x = \exp(x)$ *assuming* that e^x was its own derivative. This is different!)

Justify each of the following steps:

(a) exp(x) satisfies the following two properties:

$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y) && \text{for all } x, y > 0; \\ \exp(-x) &= \tfrac{1}{\exp(x)} && \text{for all } x. \end{aligned}$$

Hint: Let $\exp_k(x)$ *denote the* k^{th} *partial sum. To prove the first property, show that*

$$\exp_k(x+y) \leq \exp_k(x)\exp_k(y) \leq \exp_{2k}(x+y).$$

For the second, show that

$$|\exp_k(x)\exp_k(-x) - 1| \leq \sum_{n=k+1}^{2k} \frac{|2x|^n}{n!}$$

(b) Therefore $e^r \exp(r)$ for all $r \in \mathbb{Q}$.

(c) The function e^x is continuous on $\mathbb{R}$. [*Hint: You will need to use the fact that* e^x *is increasing and that* $e^{\frac{1}{n}} \to 1$ *as* $n \to \infty$*; you may take both facts as proved.*]

(d) Therefore, $e^x = \exp(x)$ for all $x \in \mathbb{R}$. QED

Yes, it really is this complicated!! There's a lot they don't tell you in calculus...