Final Exam Spr 02 Hitrik

Math 126

Problem 1 (20 points). Solve a Cauchy problem for the first order linear PDE,

$$(1) \qquad \begin{cases} \partial_x u + \partial_y u = x^2, \\ u(0,y) = y. \end{cases}$$

Problem 2 (30 points). Consider an initial–boundary value problem for the heat equation on the half-line,

$$(2) \qquad \begin{cases} (\partial_t - \partial_x^2)\, u = 0, \ \ x > 0, t > 0, \\ u(0,t) = 0 \ \ t > 0, \\ u(x,0) = \varphi(x), \ \ x > 0. \end{cases}$$

Starting from the solution formula for the initial value problem for the heat equation on the whole line, *derive* the formula for the solution of (2), $u(x,t)$.

Problem 3 (30 points). Let us consider the equation describing the propagation of waves in a one-dimensional medium with a variable speed of light $c = c(x) > 0$. The equation has the form

$$(3) \qquad \partial_x \left(c(x)^2 \partial_x u\right) = \partial_t^2 u.$$

We consider (3) on a bounded interval $[0, L]$ with the Dirichlet boundary conditions,

$$u(0,t) = u(L,t) = 0.$$

Show that the *energy* of the wave at time t,

$$E(t) := \frac{1}{2} \int_0^L \left((\partial_t u)^2 + c(x)^2 \left(\partial_x u\right)^2\right)\, dx$$

is independent of t.

Problem 4 (30 points). Consider the function $f(x) = x$ on the interval $[0, \pi]$. Find the coefficients of the Fourier cosine series of f, and show explicitly that this cosine series converges uniformly on $[0, \pi]$.

Problem 5 (30 points). Let $A = -d^2/dx^2$ acting on C^2-functions f on $[0, L]$ with the boundary conditions $f(0) = f'(L) = 0$. Show that A is symmetric with respect to the inner product $(f, g) = \int_0^L f(x)\overline{g(x)}\,dx$, and find the eigenvalues and eigenfunctions of A. Use this to compute the integral

$$\int_0^L \sin\left(\frac{(2n+1)\pi x}{2L}\right)\sin\left(\frac{(2m+1)\pi x}{2L}\right)\,dx,$$

when m and n are integers with $m \neq n$.

Problem 6 (20 points). Let D be a connected bounded open set in $\mathbf{R}^2$, and consider the Dirichlet boundary value problem for the Laplacian in D,

$$\begin{cases} \Delta u = 0 \text{ in } D, \\ u = h \text{ on } \partial D. \end{cases} \tag{4}$$

Here it is assumed that $u \in C^2(D) \cap C(\overline{D})$ and $h \in C(\partial D)$.

- (5 points) State the maximum principle for harmonic functions.
- (5 points) Explain why the problem (4) has at most one solution.
- (10 points) Can you say for sure that u is actually more regular than C^2 in D? Why?

Problem 7 (40 points). Consider the initial–boundary value problem for the heat equation,

$$\begin{cases} (\partial_t - \partial_x^2)\, u = 0, \;\; 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, \;\; t > 0, \\ u(x,0) = \varphi(x), \;\; 0 < x < L, \end{cases} \tag{5}$$

where we assume that φ is bounded and *nonnegative*, $\varphi(x) \geq 0$, $x \in (0, L)$.

- (10 points) Show that $u(x,t) \geq 0$ for $x \in (0, L)$, $t > 0$.
- (20 points) Show that $\partial_x u(0,t) \geq 0$ and $\partial_x u(L,t) \leq 0$. (Hint: Study the difference quotients.)
- (10 points) Let

$$Q(t) = \int_0^L u(x,t)\,dx$$

be the total amount of heat at time t. Show that $Q(t)$ is a decreasing function of t. (Hint: What is the derivative of $Q(t)$?)