Name: ______________________________

MATH 142: FINAL EXAM

Westin

TUESDAY, MAY 20, 2003

1.

(a) **(7 points)** Prove the *intermediate value theorem*: if $f : [a, b] \to \mathbf{R}$ is a continuous map such that $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

(b) **(3 points)** Let $f(x) = a_n x^n + \cdots a_0$ be a polynomial with real coefficients and odd degree. Prove that f has a real root; that is, prove that there is $r \in \mathbf{R}$ such that $f(r) = 0$.

2. (5 points each) Let X be the set $\mathbf{Z}$ of integers endowed with a topology in which a subset $U \subseteq X$ is open if and only if at least one of the following conditions holds:

- $X - U$ is finite;
- $0 \notin U$.

(X is called the *countable fort space*).

(a) Check that this really does define a topology on X.
(b) Prove that X is compact.
(c) Is X connected? Prove your answer.

3. (5 points each) Let d denote the usual Euclidean metric

$$d\big((x, y), (x', y')\big) = \sqrt{(x - x')^2 + (y - y')^2}$$

on $\mathbf{R}^2$. Define the *post office metric* d' on $\mathbf{R}^2$ by

$$d'\big((x, y), (x', y')\big) = \begin{cases} d\big((x, y), 0\big) + d\big((x', y'), 0\big) & (x, y) \neq (x', y') \\ 0 & (x, y) = (x', y') \end{cases}$$

(a) Prove that d' really is a metric.
(b) Describe the basic open sets

$$B_{d'}\big((x, y), r\big) := \{(x', y') \in \mathbf{R}^2 \,;\, d'\big((x, y), (x', y')\big) < r\}$$

for $(x, y) \in \mathbf{R}^2$ and $r > 0$. (Note: There are several very different cases to consider.)

4. (10 points) Let X be a Hausdorff topological space and let $A \subseteq X$ be a compact subset of X. Prove that for any $x \in X - A$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $A \subseteq V$.

5. (**5 points each**) Describe the homomorphism

$$f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$$

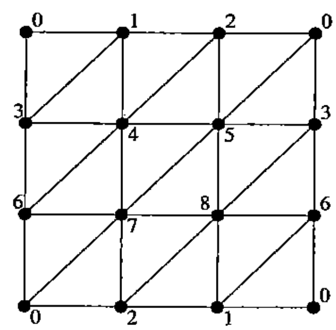
for each of the continuous maps f below. (For the first two maps we regard S^1 as the unit circle in the complex plane, while for the last map we regard it as the unit circle in $\mathbf{R}^2$.)

(a) The map $f(z) = -z$;
(b) The map $f(z) = z^n$ for fixed $n \in \mathbf{Z}$;
(c) The map

$$f(x,y) = \begin{cases} (x,y) & y \geq 0; \\ (x,-y) & y \leq 0. \end{cases}$$

6.

(a) (**10 points**) Compute the edge group of the simplicial complex K given by



(including all 18 two-simplices).

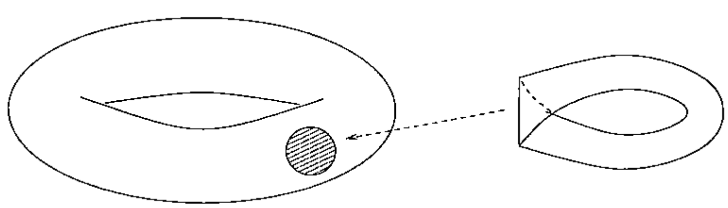
(b) (**5 points**) Let $H(p)$ and $M(q)$ denote the surfaces obtained from S^2 by adding p handles and q möbius strips, respectively. Recall that the abelianized fundamental groups of these surfaces are given by

$$\pi_1(H(p))^{\mathrm{ab}} \cong \mathbf{Z}^{2p}$$

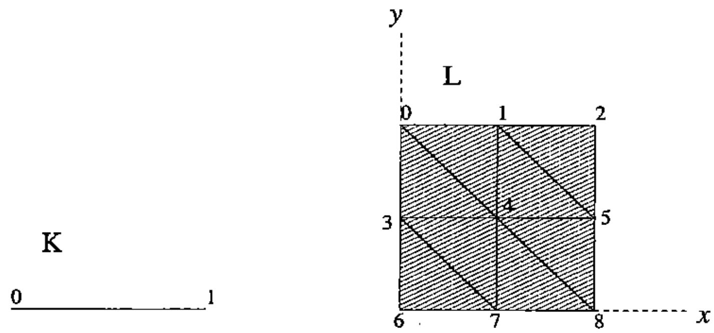
$$\pi_1(M(q))^{\mathrm{ab}} \cong \mathbf{Z}^{q-1} \times \mathbf{Z}/2\mathbf{Z}$$

Using this explain where the surface $|K|$ of (a) fits into the classification of surfaces.

7. (**12 points**) Compute the fundamental group of the space obtained from a torus by replacing a closed disk with a möbius strip glued in along its boundary circle.



8. Let K and L denote the simplicial complexes pictured below, positioned so that $|K| = [0,1] \subseteq \mathbf{R}$ and $|L| = [0,1] \times [0,1] \subseteq \mathbf{R}^2$.



Let $F : |K| \to |L|$ be the map given by $f(x) = (x, x^2)$.

(a) **(7 points)** Find an $m \geq 0$ and a simplicial approximation $f : K^m \to L$ to F. (Recall that this is a simplicial map such that $|f|(x)$ lies in the smallest simplex containing $F(x)$ for all $x \in |K|$.)

(b) **(6 points)** Construct a homotopy

$$G : |K| \times [0,1] \to |L|$$

from $|f|$ to F; that is, G should be continuous and satisfy

$$G(x,0) = |f|(x), \qquad G(x,1) = F(x)$$

for all $x \in |K|$.