MATH 142: EXAM 1

MONDAY, MARCH 3 2003 Westin

1. (10 points) Let $f : X \to Y$ be a continuous map of topological spaces with X connected. Prove that the image $f(X)$ of f is connected (where as always $f(X)$ is given a topology as a subspace of Y).

2. In (a)-(b) below you are given a subset A of a topological space X. Determine the closure of A in X. You should explain your answer, but a formal proof is unnecessary.

(a) **(5 points)** $A = \left\{\frac{a}{2^n}\,;\, a, n \in \mathbf{Z}, n \geq 0\right\}$ the set of all rational numbers with denominator a power of 2, regarded as a subspace of $X = \mathbf{R}$ with the Euclidean topology;

(b) **(5 points)** $A =$ the graph of the polar equation $r = \frac{\theta}{1+\theta}$ for $\theta > 0$ in $X = \mathbf{R}^2$ with the Euclidean topology. (Hint: $r = \theta$ is the usual polar spiral, and the graph above is simply the composition of this with $r \mapsto \frac{r}{1+r}$.)

3. Let X be a topological space and let

$$\Delta := \{(x, x) \in X \times X\,;\, x \in X\} \subseteq X \times X$$

be the diagonal. Let $f : X \to X$ be a continuous map, and let

$$\Gamma_f : X \to X \times X$$
$$x \mapsto (x, f(x))$$

be the graph of f.

(a) **(10 points)** Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) **(3 points)** Prove that

$$\Gamma_f^{-1}(\Delta) = \{x \in X\,;\, f(x) = x\}.$$

(c) **(3 points)** Prove that if X is Hausdorff, then

$$\{x \in X\,;\, f(x) = x\}$$

is a closed subset of X.

4. Let X be the topological space which as a set is simply the set of real numbers, but which has basis of open sets consisting of half-open intervals

$$[a, b) = \{x \in \mathbf{R}\,;\, a \leq x < b\}$$

for $a, b \in \mathbf{R}$, $a < b$. (Thus an arbitrary open set in X is a union of such half-open intervals.)

(a) **(5 points)** Prove that X is Hausdorff.

(b) **(4 points)** Prove that X is not connected.

(c) **(5 points)** Is the open interval $(0, 1)$ open in X? Prove your answer.