'CHOOSE YOUR OWN ADVENTURE' FINAL

MATH 115 — DECEMBER 12, 2002

Name: ...Prof. L. Bartholdi...

THE BERKELEY DEPARTMENT OF MATHEMATICS WAS THE MOST FAMOUS SCHOOL OF THE MATHEMATICS WORLD, AND HARRY POTTER WAS ITS MOST FAMOUS STUDENT. HIS MERE PRESENCE MADE SURE THAT EACH YEAR, TWENTY CANDIDATES APPLIED FOR EVERY OPEN SPOT, NO MATTER HOW RAPACIOUS BERKELEY'S TUITION BECAME. AS A RESULT, HARRY AND THE DEPARTMENT HAD COME TO AN UNSPOKEN AGREEMENT: REGARDLESS OF HIS GRADES, HARRY COULD REMAIN AT BERKELEY AS LONG AS HE WISHED. HE HAD JUST BEGUN HIS ELEVENTH YEAR. THIS ARRANGEMENT MADE STUDYING UNNECESSARY, AND TURNED EACH EVENING FROM A TIME OF FRENZIED SCHOLARSHIP TO ONE OF RELAXED COMTEMPLATION OF THE DAY'S EVENTS. THERE WAS ALSO AMPLE TIME FOR MISCHIEF.

WILL YOU HELP HARRY GET OUT OF HERE? ANSWER FOR HIM TO THE FOLLOWING QUESTIONS.

If a letter is furnished in the question, put it in the next available square above, or the square requested in the question. If the answer is a number, convert it to a letter by reducing modulo 26, and using the code

A=0	B=1	C=2	D=3	E=4	F=5	G=6	H=7	I=8
J=9	K=10	L=11	M=12	N=13	O=14	P=15	Q=16	R=17
S=18	T=19	U=20	V=21	W=22	X=23	Y=24	Z=25.	

Note that the answer will not necessarily form a word, nor even something pronounceable by a standard anglo-saxon mouth. As soon as you have filled in all the squares, you may hand in your test.

1. What is the smallest (positive) prime factor of 1001? Don't write anything down, but keep adding up all digits of the answer till you get a single digit, and go to the question with that number.

2. How many factors does 96 have? Convert to a letter and write in the next available box. Then go to question 13.

3. What is the least common multiple of $1, 2, \ldots, 10$? Convert to a letter and write in the next available box. Then go to question 13.

4. For which even n is $(n/2)!$ divisible by the least common multiple of $1, 2, \ldots, n$? If there are no such n, write a "R" in the next box. If there are finitely many such n's, write a "S" in the next box. If there are infinitely many such n's, write a "T" in the next box. Go to question 13.

5. What is the least common multiple of 12, 23 and 34? If it's above 1000, write "A" in the first box. If it's below, write "T" in the first box. Then go to question 8.

6. Compute the factorization of 54321, and save it for later use. Then go to question 9.

7. What is the greatest common divisor of 135 and 300? Write the corresponding letter in the first box. Move to question 13.

8. What is the capital of Assyria? Write the first letter in the last box. Then go to question 13.

9. What is the airspeed velocity of an unladen swallow? Write the answer in the last box. Then go to question 13.

10. Can you express ω^2 in the form $a + b\omega$? If so, go to question 16. Otherwise, go to question 27.

11. Try harder — here are some ways: $7 = F_1 + \cdots + \cdots + F_1 = F_3 + F_3 + F_3 + F_1 = F_4 + F_4 + F_1 = F_5 + F_3$. Go back to question 27.

12. This is a famous conjecture, due to Erdös and Strauss, and still unsolved. You're welcome to think about it, but it's outside the scope of this class.

13. For which n is the sum of divisors of n odd? If you think it's for squares, write a "A" in the next box. If you think it's for all proper powers, write a "E" in the next box. If you think it's for primes, write a "I" in the next box. If you think it's for all squares and doubles of squares, write a "O" in the next box. If you think it's a different answer, write a "U" in the next box. Move to question 14.

14. Consider the complex number $\omega = \frac{-1+\sqrt{-3}}{2}$. If you think it's a cube root of unity, go to question 20. If you think it's a sixth root of unity, go to question 22. If you think it's a different answer, go to question 27.

15. Does $2+\omega$ divide $5+\omega$? If yes, write "Y" in the next box; otherwise, write "N" in the next box. Go to question 17.

16. Compute $(10+\omega)^2$ in the form $a+b\omega$, and write b in the next box. Then go to question 15.

17. Among the elements $2+\omega$, $3+\omega$, $4+\omega$, $5+\omega$, how many are prime? If you think 2 or less are prime, go to question 18. If you think 3 or more are prime, go to question 19.

18. In how many prime factors does $7+2\omega$ factor? Erase the contents of the last box, and write that number. Proceed to question 19.

19. Which of the rational primes $2, 3, 5, 7, 11, 13$ are prime, as elements of A? If it's $2, 3, 5, 7$, write "F" in the next box. If it's $2, 3, 7, 11$, write "S" in the next box. Otherwise, write "T" in the next box. Go to question 33.

20. Consider the algebraic structure $A = \{a + b\omega : a, b \in \mathbb{Z}\}$. This forms a ring, meaning that there is an addition and a multiplication on A that follow the usual axioms of integers. Therefore, there is a notion of divisibility in A, and hence a notion of prime number in A, analogous to that of primes in $\mathbb{Z}$ (which are called *rational primes*).

Now, what is multiplication in A exactly? consider $(a+b\omega)(c+d\omega)$. If you think it makes $(ac-bd)+(ad+bc)\omega$, write an "N" and go to question 15. If you think it makes $(ac-bd)+(ad+bc-bd)\omega$, go to question 16. If you think otherwise, go to question 10.

21. Try harder — return to where you came from.

22. Compute ω^3. If you get -1, go to question 20. If you get something else, go to question 27.

23. What is the most compact representation of 33, i.e. the one with the smallest number of terms? Write that number of terms in the next box, and go to question 25.

24. Continued fractions are expressions of the form $a_0+1/(a_1+1/(a_2+1/(\dots)))$, usually written $[a_0; a_1, a_2, \dots]$. Consider the number x whose continued fraction is $[1; 1, 1, \dots]$, and write the integer part of $10x$ in the next box. Then go to question 36.

25. We want a way of expressing positive numbers in a unique way in "base Fibonacci" — the equivalent of restricting digits to $\{0, \dots, 9\}$ in base 10. Which of the following is the right condition? We should consider representations $n = F_{i_1} + \cdots + F_{i_\ell}$ where $i_1 \geq i_2 \geq \cdots \geq i_\ell$. Furthermore, if we should require

- all the i_j's are distinct: write "J" in the next box.
- all the i_j's are distinct, and $i_\ell \geq 2$: write "K" in the next box.
- the difference between consecutive i_j's is at least 2: write "L" in the next box.
- the difference between consecutive i_j's is at least 2, and $i_\ell \geq 2$: write "M" in the next box.
- something altogether different: write "N" in the next box.

Then go to question 33.

26. Write a table of $x^2 - 3y^2$ for $1 \leq x \leq 5$ and $1 \leq y \leq 5$. Then go to question 36.

27. Consider the Fibonacci numbers $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$. Numbers can be expressed in "base Fibonacci", i.e. as a non-increasing sum of Fibonacci numbers, just as in base 10 numbers are expressed as a sum of powers of 10. In how many ways can write 7 in "base Fibonacci"? If you can do it at least 7 ways, write that number and go to question 23. Otherwise, go to question 11.

28. What is 2^5 modulo 11? if you think it's -1, go to question 30. Otherwise, go to question 21.

29. You're almost done — here's a tough one: is it true for all $n \in \mathbb{N}$ that $4/n$ can be written as an Egyptian fraction as $4/n = 1/n_1 + 1/n_2 + 1/n_3$ and $n_1 < n_2 < n_3$? If it's false, give an $n \in \mathbb{N}$ for which it fails. If it's true, write a proof in the margin. If you want a hint, go to question 12. Otherwise take a well-deserved break.

30. How many primitive roots are there modulo 11? Write that in the next box. Find then a primitive root modulo 121. Do you know how many there are? If you know, go to the question with that number. Otherwise, go to question 31.

31. Does the equation $x^2 + x - 1$ have a solution modulo 257? If you think so, write a "L" in the next box, and go to question 35. If you think not, go to question 38. If you want a clue, go to question 39.

32. OK, you found $2^2 - 3 \cdot 1 = 1$, right? Now find a solution with $y \geq 10$, and write x in the next box. Then go to question 37.

33. OK — usual primes now. Estimate how many primes there are between 10000 and 20000. If you think there are less than 100, write "O" in the next box. If you think there are between 101 and 400, write "U" in the next box. If you think there are between 401 and 2500, write "A" in the next box. If you think there are more than 2500, write "I" in the next box. Go to question 34.

34. Computations mod p — What is the order mod 11 of 2? If you think it's 1, go to question 21. If you think it's 5, go to question 28. If you think it's 10, go to question 30.

35. What is the continued fraction of $\sqrt{3}$? If you have no clue about continued fractions, go to question 24. If you think it's $[1;1,2,3,4,\dots]$, write a "T" in the next box. If you think it's $[1;1,2,1,2,\dots]$, write a "S" in the next box. If you think it's $[1;2,1,2,1,\dots]$, write a "R" in the next box. If it's something different, write a "U" in the next box. Then continue to question 36.

36. Find the smallest solution to the equation $x^2 - 3y^2 = 1$ with positive x, y. If it has $x = y + 1$, go to question 32. Otherwise go to question 26.

37. Egyptian fractions are expressions of the form $1/n_1 + \cdots + 1/n_\ell$, with $n_1 < n_2 < \cdots < n_\ell$. Find an expression of 3/7 as an Egyptian fraction with n_ℓ minimal; write n_ℓ in the next box. Then go to question 29.

38. Well, does $x^2 + 1$ have a solution modulo 257? If yes, write a "K" in the next box, and go to question 35. Otherwise, write an "X" in the next box, and go back to question 31 with 17 in place of 257.

39. Think about quadratic reciprocity, and the standard formula to solve a degree-2 equation. Go back to question 31.

40. Yup, you're right — nevertheless, go to question 31.