Mathematics 54W

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Midterm 2

Name	
T.A	

The boxes below are for your scores, do not write in them! Write your solutions in the spaces provided after each problem.

1	9	10
2	1	15
3	15	15
4	15	15
5	9	10
Total	49	65

1. Find the numbers x and y which give the best solution (in the sense of least squares) to the inconsistent system of equations

$$\begin{aligned} 1 &= 2x + y \\ 3 &= y \\ 3 &= x + y \\ 5 &= x + y \end{aligned}$$

$A^T A x = A^T y$ 9

$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \\ 5 \end{bmatrix}$ ✓

$\begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$

$\begin{bmatrix} 6 & 4 \\ -2 & 0 \end{bmatrix}_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$ ✓

$-2x = 12$ $6x + 4y = 10$

$x = -6$ $-36 + 4y = 10$ $y = \frac{46}{4} = \frac{23}{2}$

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} & -1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$

2. Let W be the set of all $(x, y, z, w) : x + y = z + w = 0$.

(a) Find an orthogonal basis for W.

(b) Find the orthogonal projection of $(2, 0, 3, 5)$ onto W.

(c) Find the distance from $(2, 0, 3, 5)$ to W.

$w + z - x - y = 0$

$w = x + y - z$

Basis: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

Gram-Schmidt:

$u_1 = \frac{(1,0,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$

$w_2 = v_2 - (v_2 \cdot u_1) u_1 = (0,1,0,1) - \left[(0,1,0,1)\cdot\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right)$

$= (0,1,0,1) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right)$

$u_2 = \frac{\left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$

$= \left(-\frac{\sqrt{2}}{2}, \sqrt{2}, 0, \frac{\sqrt{2}}{2}\right)$

$u_2 = \left(-\frac{1}{\sqrt{2}}, \sqrt{2}, 0, \frac{1}{\sqrt{2}}\right)$

$w_3 = (0,0,1,-1) - \left[(0,0,1,-1)\cdot\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right)\right]u_1 - \left[(0,0,1,-1)\cdot\left(-\frac{1}{\sqrt{2}},\sqrt{2},0,\frac{1}{\sqrt{2}}\right)\right]u_2$

$= (0,0,1,-1) - \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}},\sqrt{2},0,\frac{1}{\sqrt{2}}\right)$ u_2

$= \left(0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}, 0 + 1, 1, -1 + \frac{1}{2} + \frac{1}{2}\right) = (0,1,1,0)$

$u_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$w_4 = (0,0,0,1) - \left[(0,0,0,1)\cdot u_1\right]u_1 - \left[v_4 \cdot u_2\right]u_2 - \left[v_4 \cdot u_3\right]u_3$

$= \left(0 - \frac{1}{2} + \frac{1}{2} - 0, 0 - 0 - 1 - 0, 0 - 0 - 0 - 0, 1 - \frac{1}{2} - \frac{1}{2} - 0\right)$

$u_4 = (0, -1, 0, 0)$

3. Let $A = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

$\det(A-\lambda I) = \det\begin{bmatrix} 6-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (6-\lambda)(2-\lambda)-12$

$= 12 - 8\lambda + \lambda^2 - 12$

$= \lambda(\lambda - 8)$

(a) Find the eigenvalues of A.

$\lambda = \{0, 8\}$ ✓

(b) We know that there exist matrices S and T such that $A = STS^{-1}$, where T is upper triangular and S is invertible. Find T.

$\lambda = 0 \quad NS(A-\lambda I) = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 6 & 3 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 8 \quad \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$T = \begin{bmatrix} 0 & \\ & 8 \end{bmatrix}$ ✓

$T = \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

(c) Now find S.

$S = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$ ✓

4. Let $A = \begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix}$.

$\det(A-\lambda I) = \det\begin{bmatrix} 6-\lambda & -1 \\ 4 & 2-\lambda \end{bmatrix} = 12 - 8\lambda + \lambda^2 + 4$

(a) Find the eigenvalues of A.

$\lambda = 4$ ✓

$\lambda^2 - 8\lambda + 16 = (\lambda-4)^2$

(b) We know that there exist matrices S and T such that $A = STS^{-1}$, where T is upper triangular. Find T.

$\lambda = 4 \quad NS(A-\lambda I) = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$NS(A-\lambda I) = 1$
we need
$NS = 2$ so
use Jordan
format

$T = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$ ✓

$NS = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c) Now find S.

$S = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$

Since 2^{nd} column of $(A-\lambda I)$ is nonzero, $s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$s_1 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} s_2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$S = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$ ✓

5. For each of the following matrices A, you are asked to determine whether or not there exist matrices S and D such that $A = SDS^{-1}$, with S invertible and then with S orthogonal. In the appropriate place in the box below, write Y if S and D with the properties named exist, N if not, and M if there is not enough information to tell.

(a)

S is basis

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

det(A): 6 ≠ 0
A invertible

(b)

S is basis

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

det(A): 6

(c)

$\lambda : 1 \quad \begin{bmatrix} 0 & & \\ 1 & 0 & \\ 3 & 3 & 2 \end{bmatrix}$

NS = 2

λ: 1, 3

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

det (A): 3

(d)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

symmetric
A is diagonalizable $A = S\Lambda S^{-1}$

(e) A is a matrix whose characteristic polynomial is $X(X-1)(X-2)$.

λ: 0, 1, 2

S is basis

det A = 0

d) $\lambda^2 - tr(A)\lambda + det(A)$

$\lambda^2 - 9\lambda + (15 + 24 + 24 - 27 - 16 - 20)$

+2

	Invertible S	Orthogonal S
a	Yes	No
b	Yes	No
c	~~Yes~~ N	No
d	Yes	Yes
e	No	No

for orthogonality of S A must be symm and have diff eigenvalues