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First Midterm Examination
Monday February 23 2004
Closed Books and Closed Notes

Question 1
Constraint Forces and Friction

Consider a particle which is in motion on a rough surface. A curvilinear coordinate system q^1, q^2, q^3 is chosen such that the surface can be described by the equation

$$q^3 = d(t), \tag{1}$$

where $d(t)$ is a known function of time t.

(a) *(10 Points)* Suppose that the particle is moving on the rough surface.

(i) Argue that $\mathbf{v}_{rel} = \dot{q}^1\mathbf{a}_1 + \dot{q}^2\mathbf{a}_2$.

(ii) Give a prescription for the constraint force acting on the particle.

(b) *(10 Points)* Suppose that the particle is stationary on the rough surface. In this case, two equivalent prescriptions for the constraint force are

$$\mathbf{F}_c = \mathbf{N} + \mathbf{F}_f = \sum_{i=1}^{3} \lambda_i \mathbf{a}^i. \tag{2}$$

(i) Show that

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & 1 \end{bmatrix} \begin{bmatrix} F_f^1 \\ F_f^2 \\ N \end{bmatrix}, \tag{3}$$

where N, and F_f^1 and F_f^2 uniquely define the normal force $\mathbf{N}$ and friction force $\mathbf{F}_f$, respectively, and

$$a_{ik} = \mathbf{a}_i \cdot \mathbf{a}_k, \qquad (i, k = 1, 2, 3)\,. \tag{4}$$

(ii) For which coordinate systems does $F_f^1 = \lambda_1$, $F_f^2 = \lambda_2$ and $N = \lambda_3$? Give an example to illustrate your answer.

(c) *(5 Points)* Suppose that a spring force and a gravitational force also act on the particle. Prove that the total energy of the particle is not conserved, even when the friction force is static.

Question 2

A Particle Moving on a Helicoid

Consider a particle of mass m which is in motion on a helicoid. In terms of the cylindrical polar coordinates r, θ, z, the equation of the right helicoid is

$$z = \alpha\theta, \tag{5}$$

where α is a constant. A gravitational force $-mg\mathbf{E}_3$ acts on the particle.

(**a**) *(10 Points)* Consider the following curvilinear coordinate system for $\mathcal{E}^3$:

$$q^1 = \theta, \quad q^2 = r, \quad q^3 = \nu = z - \alpha\theta. \tag{6}$$

Either show that

$$\mathbf{a}_1 = r\mathbf{e}_\theta + \alpha\mathbf{E}_3, \quad \mathbf{a}_2 = \mathbf{e}_r, \quad \mathbf{a}_3 = \mathbf{E}_3, \tag{7}$$

or show that

$$\mathbf{a}^1 = \frac{1}{r}\mathbf{e}_\theta, \quad \mathbf{a}^2 = \mathbf{e}_r, \quad \mathbf{a}^3 = \mathbf{E}_3 - \frac{\alpha}{r}\mathbf{e}_\theta. \tag{8}$$

(**b**) *(15 Points)* Consider a particle moving on the smooth helicoid:

(i) What is the constraint on the motion of the particle, and what is a prescription for the constraint force $\mathbf{F}_c$ enforcing this constraint?

(ii) Show that the equations governing the unconstrained motion of the particle are

$$\begin{aligned} \frac{d}{dt}\left(m\left(r^2 + \alpha^2\right)\dot{\theta}\right) &= -mg\alpha, \\ \frac{d}{dt}\left(m\dot{r}\right) - mr\dot{\theta}^2 &= 0. \end{aligned} \tag{9}$$

(iii) Prove that the angular momentum $\mathbf{H}_O \cdot \mathbf{E}_3$ is not conserved.

(**c**) *(10 Points)* Suppose the non-integrable constraint

$$r\dot{\theta} + h(t) = 0, \tag{10}$$

is imposed on the particle. Establish a second-order differential equation for $r(t)$, a differential equation for θ and an equation for the constraint force enforcing the non-integrable constraint. Indicate how you would solve these equations to determine the motion of the particle and the constraint forces acting on it.

Notes on Cylindrical Polar Coordinates

Recall that the cylindrical polar coordinates $\{r, \theta, z\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$r = \sqrt{x_1^2 + x_2^2}\,, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right)\,, \quad z = x_3\,.$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

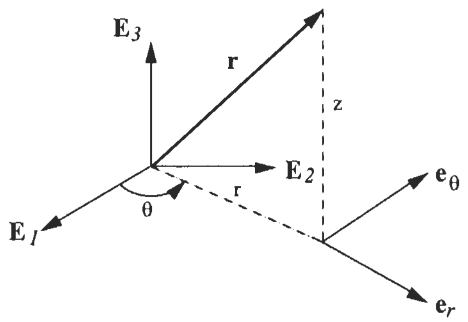


Figure 1: Cylindrical polar coordinates

For the coordinate system $\{r, \theta, z\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_r\,, \quad \mathbf{a}_2 = r\mathbf{e}_\theta\,, \quad \mathbf{a}_3 = \mathbf{e}_z.$$

In addition, the contravariant basis vectors are

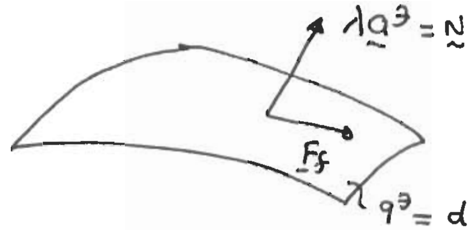
$$\mathbf{a}^1 = \mathbf{e}_r\,, \quad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta\,, \quad \mathbf{a}^3 = \mathbf{e}_z.$$

The gradient of a function $u(r, \theta, z)$ has the representation

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{\partial u}{\partial \theta}\frac{1}{r}\mathbf{e}_\theta + \frac{\partial u}{\partial z}\mathbf{E}_3.$$

Question 1

$\{q^i\}$ coordinate system.



(a)

(i) The velocity vector of the particle is $\underset{\sim}{v} = \dot{q}^1 \underline{a}_1 + \dot{q}^2 \underline{a}_2 + \dot{q}^3 \underline{a}_3$

However because the particle is moving on the surface $\dot{q}^3 = \dot{d}$. Further, if the surface was fixed $\dot{d} = 0$. Hence,

$$\underline{v}_{rel} = \underset{\sim}{v}\Big|_{\text{surface is fixed}} = \dot{q}^1 \underline{a}_1 + \dot{q}^2 \underline{a}_2$$

(ii)

$$\underline{F}_c = \lambda \underline{a}^3 + \underline{F}_f \qquad \underline{F}_f = -\mu_d \|\underset{\sim}{N}\| \frac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|} \qquad \underset{\sim}{N} = \lambda \underset{\sim}{a}^3$$

(b) $$\underline{F}_c = \underset{\sim}{N} + \underline{F}_f = \sum_{i=1}^{3} \lambda_i \underset{\sim}{a}^i$$

(i) Now

$$\lambda_i = \underline{F}_c \cdot \underline{a}_i = \underline{N} \cdot \underline{a}_i + \underset{\sim}{F}_f \cdot \underline{a}_i$$

But $\underline{F}_f = F_f^1 \underline{a}_1 + F_f^2 \underline{a}_2$, $\quad \underline{N} = N \underset{\sim}{a}^3$

Hence

$$\lambda_1 = F_f^1 \underline{a}_1 \cdot \underline{a}_1 + F_f^2 \underline{a}_1 \cdot \underline{a}_2 + N \underline{a}^3 \cdot \underset{\sim}{a}_1$$

$$= a_{11} F_f^1 + a_{12} F_f^2 \qquad \text{where} \quad a_{ik} = \underline{a}_i \cdot \underline{a}_k$$

Similarly

$$\lambda_2 = a_{12} F_f^1 + a_{22} F_f^2$$

and

$$\lambda_3 = a_{13} F_f^1 + a_{23} F_f^2 + N$$

(ii) For $F_f^1 = \lambda_1$ and $F_f^2 = \lambda_2$ we need $a_{11} = 1$, $a_{12} = 0$, $a_{22} =$

For $N = \lambda_3$ we need $a_{13} = 0$, $a_{23} = 0$, $a_{33} = 1$

Hence $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ = orthonormal basis. Restrictions on q^i are $\frac{\partial \underline{r}}{\partial q^i}$ are orthonormal

For example: Cartesian coordinates – particle is moving on a plane.

(iii)

$$\frac{dT}{dt} = \underline{F} \cdot \underline{v} = (\underline{F}_s - mg\underline{E}_3) \cdot \underline{v} + \underline{F}_c \cdot \underline{v}$$

$$= -\frac{d}{dt}\left(U_s + mg \underline{E}_3 \cdot \underline{r} \right) + \underline{F}_c \cdot \underline{v}$$

$$\Rightarrow \frac{d}{dt}\left(E = T + U_s + mg \underline{E}_3 \cdot \underline{r} \right) = \underline{F}_c \cdot \underline{v} = \lambda \dot{d} + \underline{F}_f \cdot \underline{v} \neq 0 \text{ even when } \underset{\sim}{v}_{rel} = 0.$$

alternatively

$$\frac{dE}{dt} = \underset{\sim}{F}_{nc} \cdot \underline{v} = (\underline{N} + \underline{F}_f) \cdot \underline{v}$$

$$= \left(\sum_{i=1}^{3} \lambda_i \underline{a}^i\right) \cdot \left(\sum_{k=1}^{3} \dot{q}^k \underline{a}_k\right)$$

$$= \lambda_1 \dot{q}^1 + \lambda_2 \dot{q}^2 + \lambda_3 \dot{q}^3$$

$$= \lambda_1 \dot{q}^1 + \lambda_2 \dot{q}^2 + \lambda_3 \dot{d}$$

When particle is not moving relative to the surface, this reduces to

$$\dot{E} = \lambda_3 \dot{d}$$

which is not necessarily zero.

Question 2

(a) $q^1 = \theta, \quad q^2 = r, \quad q^3 = \upsilon = z - \alpha\theta$

$$\underline{r} = r\underline{e}_r + (\upsilon + \alpha\theta)\underline{E}_3$$

(i) $\underline{a}_1 = \dfrac{\partial \underline{r}}{\partial \theta} = r\underline{e}_\theta + \alpha\underline{E}_3, \quad \underline{a}_2 = \dfrac{\partial \underline{r}}{\partial r} = \underline{e}_r, \quad \underline{a}_3 = \dfrac{\partial \underline{r}}{\partial \upsilon} = \underline{E}_3$

(ii) In cylindrical coordinates $\nabla u = \dfrac{\partial u}{\partial r}\underline{e}_r + \dfrac{\partial u}{\partial \theta}\dfrac{1}{r}\underline{e}_\theta + \dfrac{\partial u}{\partial z}\underline{E}_3$

$\underline{a}^1 = \operatorname{grad}\theta = \frac{1}{r}\underline{e}_\theta, \quad \underline{a}^2 = \operatorname{grad} r = \underline{e}_r, \quad \underline{a}^3 = \operatorname{grad}\upsilon = \underline{E}_3 - \frac{\alpha}{r}\underline{e}_\theta$

(b) Particle on Helicoid

(i) Constraint $\psi(\underline{r}, t) = \upsilon = 0$

Constraint force $\underline{F}_c = \lambda\underline{a}^3 = \lambda\left(\underline{E}_3 - \frac{\alpha}{r}\underline{e}_\theta\right)$

(ii) For this coordinate system

$$T = \tfrac{1}{2}m\underline{v}\cdot\underline{v} = \tfrac{1}{2}m(\dot\theta\underline{a}_1 + \dot r\underline{a}_2 + \dot\upsilon\underline{a}_3)\cdot\underline{v}$$
$$= \tfrac{1}{2}m((r^2+\alpha^2)\dot\theta^2 + \dot r^2 + \dot\upsilon^2 + 2\dot\upsilon\dot\theta\alpha)$$
$$\underline{F} = \lambda\underline{a}^3 - mg\underline{E}_3$$

Equations governing motion of the particle are

$$\frac{d}{dt}\left(\frac{\partial\tilde T}{\partial\dot\theta} = m(r^2+\alpha^2)\dot\theta\right) - \left(\frac{\partial\tilde T}{\partial\theta} = 0\right) = \underline{F}\cdot\tilde{\underline{a}}_1 = -mg\alpha$$

$$\frac{d}{dt}\left(\frac{\partial\tilde T}{\partial\dot r} = m\dot r\right) - \left(\frac{\partial\tilde T}{\partial r} = mr\dot\theta^2\right) = \underline{F}\cdot\tilde{\underline{a}}_2 = \underline{F}\cdot\underline{e}_r = 0$$

(iii) $\dfrac{d}{dt}\left(\underline{H}_0\cdot\underline{E}_3\right) = \dot{\underline{H}}_0\cdot\underline{E}_3 = (\underline{r}\times\underline{F})\cdot\underline{E}_3$

$$= (\underline{r}\times(-mg\underline{E}_3 + \lambda\underline{a}^3))\cdot\underline{E}_3 = (\underline{r}\times\lambda\underline{a}^3)\cdot\underline{E}_3$$
$$= \lambda\underline{a}^3\cdot(\underline{E}_3\times\underline{r}) = r\underline{e}_\theta\cdot\lambda\underline{a}^3$$
$$= -\alpha\lambda \neq 0 \Rightarrow \underline{H}_0\cdot\underline{E}_3 \text{ is not conserved.}$$

Note that $\underline{H}_0\cdot\underline{E}_3 = mr^2\dot\theta \neq \dfrac{\partial T}{\partial\dot\theta}$

(c) Constraint $r\dot\theta = -h$ (This constraint is non-integrable).

This constraint can be written as $(r\underset{\sim}{a}^1)\cdot\underline{v} + h = 0$

Constraint force is prescribed by normality $\lambda_2 \, r\underset{\sim}{a}^1 = \lambda_2 \underline{E}_\theta$

Hence $\underset{\sim}{F} = -mg\underline{E}_3 + \lambda\underset{\sim}{a}^3 + \lambda_2 r\underline{a}^1$

Eqns of motion are

$$\frac{d}{dt}\left(m(r^2+\alpha^2)\dot\theta\right) = -mg\alpha + \underline{F}_c\cdot\underline{a}_1 = -mg\alpha + \lambda_2 r$$

$$\frac{d}{dt}\left(m\dot r\right) - mr\dot\theta^2 = \underline{F}\cdot\underline{a}_2 = 0$$

$$r\dot\theta + h = 0$$

Solving these equations.

$$m\ddot r - mr\left(\frac{h}{r}\right)^2 = 0 \qquad (*)$$

$$\dot\theta = -\frac{h}{r} \qquad (**)$$

$$\lambda_2 = \frac{mg\alpha}{r} - \frac{d}{dt}\left(m(r^2+\alpha^2)\dot\theta\right) \qquad (***)$$

(*) can be solved for $r(t)$ and the result used in (**) to determine $\theta(t)$. Then (***) can be solved for $\lambda_2(t)$.