Exam 2
(Solutions)

1.) $\boxed{\text{Yes}}$ A is a 21×25 matrix. There are exactly 5 linearly independent solutions to $A\vec{x} = \vec{0}$ $\Rightarrow$ dim null $A = 5$. By the rank theorem, rank $A = 20$. Since there are 21 rows, A is not onto. Thus, $\exists$ b s.t. $A\vec{x} = \vec{b}$ has no solution.

2) $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \end{pmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ -2 & 0 & -\lambda \end{vmatrix}$

$= -\lambda(-\lambda(3-\lambda)) + 2(2(3-\lambda))$

$= \lambda^2(3-\lambda) + 4(3-\lambda) = (\lambda^2+4)(3-\lambda) \Rightarrow \lambda = 3, \pm 2i$

$\underline{\lambda = 3}$

$\begin{pmatrix} -3 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{13}{3} \end{pmatrix} \Rightarrow x_1 = x_3 = 0, \ x_2 = \text{free}$ $\quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -2i$

$\begin{pmatrix} +2i & 0 & 2 \\ 0 & 3+2i & 0 \\ -2 & 0 & 2i \end{pmatrix} \Rightarrow \begin{pmatrix} 2i & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2 = 0 \quad 2ix_1 = -2x_3$ $\quad \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$

Since coefficients of A are real: for $\lambda = +2i$ $\vec{v} = \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$, the conjugate

$P = \begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}$

$AP = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 0 \\ 0 & -2i & 2i \end{pmatrix}$

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$PD = \begin{pmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 0 \\ 0 & -2i & 2i \end{pmatrix}$ ✓

3) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \sqrt{2} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{v}_1 = \boxed{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}}, \quad \vec{v}_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}}$

$\text{Proj}_{v_1} v_3 = \frac{1}{1}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} - 2/\sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$ $\quad \text{Proj}_{v_2} v_3 = \frac{1}{1}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$

$\|v_1\| = \|v_2\| = \|v_3\| = 1$ so orthonormal already

$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = QR, \quad Q^T = Q$ by inspection

$\Rightarrow R = Q^T A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

4) $(2,6): \; B_0 + B_1 2 = 6$

$(3,7): \; B_0 + B_1 3 = 7$

$(5,11): \; B_0 + B_1 5 = 11$

$(8,20): \; B_0 + B_1 8 = 20$

$\Rightarrow \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 8 \end{pmatrix}}_{A} \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ 7 \\ 11 \\ 20 \end{pmatrix}}_{\vec{b}}$

normal eq' $\quad A^T A \vec{x} = A^T \vec{b}$

$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 11 \\ 20 \end{pmatrix} = \begin{pmatrix} 44 \\ 248 \end{pmatrix}$

$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 18 \\ 18 & 102 \end{pmatrix}$

$[A^TA \,|\, A^T\vec{b}] = \begin{bmatrix} 4 & 18 & 44 \\ 18 & 102 & 248 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{2} & 11 \\ 9 & 51 & 124 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{2} & 11 \\ 9 & 51 & 124 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & \frac{9}{2} & 11 \\ 0 & \frac{21}{2} & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{2} & 11 \\ 0 & 1 & \frac{50}{21} \end{bmatrix} \Rightarrow B_1 = \frac{50}{21}$

$B_0 = 11 - \frac{9}{2}\left(\frac{50}{21}\right) = \frac{231 - 225}{21} = \frac{6}{21} = \frac{2}{7} \Rightarrow B_0 = \frac{2}{7}$

$\Rightarrow \boxed{y = \frac{50}{21}x + \frac{2}{7}}$