Midterm 1

Name	
T.A	

Sec 106

The boxes below are for your scores, do not write in them! Write your solutions in the spaces provided after each problem. Explain your reasoning in all cases: you may be graded on your explanations as well as on your answers.

1	20	20
2	20	20
3	14	15
4	15	15
Total	69	A-

1. Find X, if possible. If not, explain why not.

(a) $\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix} = X + \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$3 = X_{11} + 0 \qquad X_{11} = 3$

$2 = X_{12} + 1 \qquad X_{12} = 1$

$5 = X_{21} + 1 \qquad X_{21} = 4$

$3 = X_{22} + 2 \qquad X_{22} = 1$

$\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

(b) $\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix} = X \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$\begin{bmatrix} X_{11}(0) + X_{12}(1) & X_{11}(1) + X_{12}(2) \\ X_{21}(0) + X_{22}(1) & X_{21}(1) + X_{22}(2) \end{bmatrix}$

$X_{11} = -4, \quad X_{12} = 3, \quad X_{22} = 5, \quad X_{21} = -7$

$\begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix}$

(c) $\mathbf{X} = (1,2,3) \cdot (-1,3,4)$ $= 1(-1) + 2(3) + 3(4)$

$= -1 + 6 + 12 = 17$

(d) $X \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 6 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

X is 2x2 matrix

$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

~~Not possible to find X because bottom row of resultant matrix is unattainable.~~

2. Consider the matrices:

$$A := \begin{pmatrix} 1 & -1 & 1 & -2 & -3 \\ 1 & 0 & 1 & -1 & -4 \\ 2 & -2 & 2 & -3 & -5 \\ 3 & -2 & 3 & -4 & -9 \end{pmatrix} \qquad \tilde{A} := \begin{pmatrix} 1 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Assuming these are row equivalent:

(a) Find a basis for the row space of A from among the rows of $\tilde{A}$.

$[1\ 0\ 1\ 0\ -3]$,
$[0\ 1\ 0\ 1\ -1]$,
$[0\ 0\ 0\ 1\ 1]$

(b) Find a basis for the column space of A from among the colums of A.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -3 \\ -4 \end{bmatrix}$$

(c) Find a basis for the null space of A.

$\tilde{A}X = 0$ free variables: $x_3 = s$, $x_5 = t$

r_3: $x_4 + x_5 = 0$ $\quad x_4 = -t$

r_2: $x_2 + x_4 - x_5 = 0$ $\quad x_2 = 2t$

r_1: $x_1 - x_3 - 3x_5 = 0$ $\quad x_1 = s + 3t$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(d) What is the rank of A?

$rk(A) = CS(A) = RS(A) = 3$

$rk(A) + NS(A) = 5$

$NS(A) = 2$

3. In each of the following examples, you are given a sequence of vectors in a vector space V. Answer the question, explaining your answer clearly, using complete sentences. Full credit will not be given if you just answer yes or no.

(a) Does the sequence $((2,2,-1,4),(1,7,3,2),(1,4,3,-1))$, form a basis for the vector space $V = \mathbf{R}^4$?

(b) Does the sequence $(x^2-2x+2, x^2+2x, x^2-1, x^2-3x+5)$ form a basis for the space of polynomials of degree less than or equal to 2?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & -3 \\ 2 & 0 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & -2 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & 0 & -2 & \frac{5}{2} \end{bmatrix}$$

(c) Suppose that (v_1, v_2, v_3, v_4) spans the null space of a 5 by 7 matrix A of rank 4. Is the sequence (v_1, v_2, v_3, v_4) linearly independent?

why ?

4/5

a. NO, because 3 vectors in $\mathbb{R}^4$ will not span the entire vector space, and a basis requires this as well as linear independence of the vectors.

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b. No, because there are 4 components and the dimension of the space (considering the degree of poly. ≤ 2) is 3. Only 3 vectors are needed to form the basis. As seen from the row echelon form above, if polynomial #4 is omitted a basis of the space is formed from the remaining polynomials. (Now they are linearly independent.)

5/5

c. $rk(A) + NS(A) = n = 7$ $\quad rk(A) = 4 \quad NS(A) = 3$

If the dimension of the null space is 3, then only 3 vectors are needed to span the Null space. Since we have 4 vectors V_1, V_2, V_3, V_4, we must have dependent set of vectors.

4. Let $M_{2,2}$ denote the vector space of all 2×2 matrices, and if $A \in M_{2,2}$, let $tr(A)$ denote the sum of the diagonal terms, and let W be the set of all $A \in M_{2,2}$ such that $tr(A) = 0$.

(a) **Find a basis for W.** Test for linear independence

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A_1x_1 + \dots + A_nx_n = 0$ x_i is a coefficient $x_i = 0$ because A_i cannot be in the span of $A_1 \dots A_n$

$W: \{ A \in M_{2,2} \mid tr(A) = 0 \}$

$a_{11} + a_{22} = 0$ $a_{11} = -a_{22}$

$$\overset{A_1}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \overset{A_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{A_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$$

$A_1 \cdots A_n$ must span W

(b) **Compute the dimension of W.**

$\dim(W) = 3$ (3 entries are needed to determine W)

↓

a_{11} + a_{22} are equal and opposite so they count as 1 entry (for basis)

$\dim(A) = 4$ (4 entries)

(c) **Find the coordinates of $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ with respect to your basis.**

$0A_1 + 2A_2 + 3A_3$