Mathematics 55 Professor K. A. Ribet
Final Examination December 17, 1997

International House
12:30–3:30 PM

Your Name: ______________________________ TA: ________________________

Please check that you have all 10 pages of this exam booklet. Write your name on *each* page. Before beginning your work, look over the whole exam to spot problems that you can do quickly. The rules are the same as for previous exams:

> You need not simplify your answers unless you are specifically asked to do so. Answers may include factorials and binomial coefficients $C(n, r)$. It is essential to write legibly and *show your work*. If your work is absent or illegible, and your answer is not perfectly correct, then no partial credit can be awarded. Completely correct answers which are given without justification may receive little or no credit.

During this exam, you are not allowed to use calculators or consult your notes or books.

Problem	Maximum	Your Score
1	10	
2	10	
3	9	
4	8	
5	9	
6	9	
7	10	
8	8	
9	9	
Total	**82**	

At the conclusion of the exam, hand in this exam paper to your TA.

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1a *(5 points)***.** How many subsets of the set $\{1, 2, \ldots, 19\}$ contain at least one odd integer?

1b *(5 points)***.** Find the last (rightmost) decimal digit of 3^{1024}.

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2a *(5 points)***.** If $P(A) = 1/3$, $P(B) = 1/2$ and $P(A \cup B) = 2/3$, find $P\left((A \cap B)|A\right)$.

2b *(5 points)***.** Mumford has a trick quarter with two heads, a trick quarter with two tails, and a standard quarter with one head and one tail. He chooses one of the three coins at random, tosses it in the air, and slaps it on the table. A head is showing. Find the probability that the selected coin has two heads.

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3 *(9 points)***.** For each positive integer n, let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ which contain no two consecutive integers.

(a) Calculate $f(1)$, $f(2)$ and $f(3)$.

(b) Find a recursive formula for $f(n)$ and use it to calculate $f(8)$. [Justify your recursive formula.]

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4 *(8 points)*. According to `math.berkeley.edu`, the number of Math 55 students who visited the class Web page at least once in November is divisible by 1, 2, 3, 4, 5 and 6, but leaves remainder 1 on division by 7. What is this number?

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5 *(9 points)*. Show that

$$\sum_{i=1}^{2n} (-1)^{i+1} \frac{1}{i} = \sum_{i=n+1}^{2n} \frac{1}{i}$$

for all positive integers n. (For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{1999} - \frac{1}{2000} = \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{2000}$.)

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6 *(9 points)*. A box contains three red socks, three blue socks, and four white socks. (Socks of the same color are indistinguishable.) Eight socks are pulled out of the box, one at a time. In how many ways can this be done?

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7 *(10 points)*. Let G be the simple graph whose vertices are the bit strings of length 6, two bit strings being connected by an edge if and only if they differ in exactly one place.

(a) Does G have an Euler circuit?

(b) Is the graph G planar?

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8 *(8 points)*. Suppose that A is a finite set with at least two elements and that R is an equivalence relation on A (for instance, the relation of acquaintance among guests at a party). Show that there are distinct elements a and a' in A whose equivalence classes $[a]_R$ and $[a']_R$ have the same number of elements.

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9 *(9 points)*. I roll a single die repeatedly, until three different faces have come up at least once. What is the expected number of times that I need to roll the die?