Mathematics 55 Professor K. A. Ribet
Final Examination May 18, 1996

155 Dwinelle
12:30–3:30 PM

Your Name: Ian Wilkes TA: ____________

Please check that you have all 10 pages of this exam booklet. Write your name on *each* page. Before beginning your work, look over the whole exam to spot problems that you can do quickly. The rules are the same as for previous exams:

> You need not simplify your answers unless you are specifically asked to do so. You need not convert binomial coefficients into quotients of factorials. It is essential to write legibly and *show your work*. If your work is absent or illegible, and your answer is not perfectly correct, then no partial credit can be awarded. Completely correct answers which are given without justification may receive little or no credit.

During this exam, you are not allowed to use calculators or consult your notes or books.

Problem	Maximum	Your Score
1	11	
2	8	
3	6	
4	9	
5	9	
6	8	
7	7	
8	10	
9	7	
Total	**75**	

At the conclusion of the exam, hand in this exam paper to your TA.

Your Name: Ian Wilkes

1a *(6 points)*. How many onto functions are there from a set with 7 elements to one with 3 elements?

1b *(5 points)*. What is the coefficient of a^7b^5 in the expansion of $(2a-b)^{12}$?

Your Name: Ian Wilkes

2 *(8 points)*. Pokerhontas drops the King of ♣, the King of ♡ and the Ace of ♠ into a large bag. She shakes the bag vigorously and then removes two of the cards without looking at them. (a) If it is known that one of the two cards is a king, what is the probability that both cards are kings? (b) If it is known that one of the two cards is the King of ♡, what is the probability that both cards are kings?

Your Name: Ian Wilkes

3 *(6 points)***.** Let G be the multigraph with adjacency matrix $\begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$. How many vertices does G have? How many edges? Does G have an Euler circuit?

Your Name: Ian Wilkes

4 *(9 points)*. Each box of C^{++} Cereal comes packed with a small plastic toy, which is shaped either like a "C" or like a "+." A given box contains a "+" with probability 2/3 and a "C" with probability 1/3. Suppose that I buy n boxes of cereal ($n \geq 3$). What is the probability that I have at least one "C" and at least two "+"s?

Your Name: Ian Wilkes

5 *(9 points)*. How many ways can n books be placed on k distinguishable shelves (a) if the books are indistinguishable copies of the same title? (b) if no two books are the same and the positions of the books on the shelves matter?

Your Name: Ian Wilkes

6 *(8 points)*. Consider the first 250 Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2, \ldots,$ $f_{249} \approx 5 \times 10^{51}$. (a) Show that there are at least 84 of them which have the same remainder when divided by 3. (b) How many of them are divisible by 2?

Your Name: Ian Wilkes

7 *(7 points)*. Find an integer d such that $(M^{11})^d \equiv M \bmod 55$ for all M prime to 55.

Your Name: Ian Wilkes

8a *(3 points)*. Show that $\dfrac{1}{\sqrt{n+1}} \geq 2\left(\sqrt{n+2}-\sqrt{n+1}\right)$ for all integers $n \geq 1$. (Multiply both sides by the positive quantity $\sqrt{n+2}+\sqrt{n+1}$.)

8b *(7 points)*. Use mathematical induction and the result of part (a) to show:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \;\geq\; 2(\sqrt{n+1}-1)$$

for $n \geq 1$.

Your Name: Ian Wilkes

9 *(7 points)***.** The diagram below, copied from a famous number theory book, shows the logical relation among chapters—to understand chapter 12, for instance, you have to have read the first five chapters as well as chapters 7 and 10. The diagram defines a partial ordering on the set $S := \{1, 2, \ldots, 15\}$: $a \prec b$ if and only if a must be read before b. With respect to this partial ordering: (a) What is the least upper bound of 2 and 8? (b) What are the maximal and minimal elements of S? Finally, (c) describe a total ordering on S which is compatible with the given partial ordering.

$$
\begin{tikzcd}
 & & 14 & & 15 \\
 & & & 13 \arrow[ul] \arrow[ur] & \\
6 & & 12 \arrow[ur] & & 11 \arrow[ul] \\
 & 5 \arrow[ul] \arrow[ur] & & & 9 \arrow[u] \\
 & 4 \arrow[u] & 10 \arrow[uu] & & 8 \arrow[u] \\
 & 3 \arrow[u] & & 7 \arrow[ul] \arrow[ur] & \\
 & 2 \arrow[u] & & & \\
 & 1 \arrow[u] & & &
\end{tikzcd}
$$