Math 55: Second Midterm — Solutions

Problem 1 (15 points):
(a) How many integers between 1 and 1000 inclusive are divisible by either 4 or 21?

Since 4 and 21 are relatively prime, $\text{lcm}(4,21) = 4 \cdot 21 = 84$. Thus numbers divisible by both 4 and 21 are precisely those divisible by 84. Since the number of integers between 1 and n inclusive which are divisible by p is $\lfloor n/p \rfloor$, the answer is

$$\lfloor 1000/4 \rfloor + \lfloor 1000/21 \rfloor - \lfloor 1000/84 \rfloor = 250 + 47 - 11 = 286$$

by inclusion-exclusion.

(b) How many integers between 1 and 1000 inclusive are divisible by either 4 or 6?

Since $\text{lcm}(4,6) = 12$, numbers divisible by both 4 and 6 are precisely those divisible by 12. Thus the answer is

$$\lfloor 1000/4 \rfloor + \lfloor 1000/6 \rfloor - \lfloor 1000/12 \rfloor = 250 + 166 - 83 = 333.$$

Problem 2 (15 points): Assume n and m are integers with $n \geq m \geq 0$. How many ways are there to put n *identical* objects into m numbered boxes in such a way that no box is empty?

Let x_i be the number of objects in box number i, for $i = 1, \ldots, m$. Then we are asking for the number of solutions $(x_1, x_2, \ldots, x_m)$ to

$$x_1 + x_2 + \cdots + x_m = n, \qquad x_i \geq 1.$$

Begin by putting 1 object in each box; then we have $n - m$ objects left to distribute among m boxes. Mathematically, put $x_i = 1 + y_i$ where $(y_1, y_2, \ldots, y_m)$ is a solution to

$$y_1 + y_2 + \cdots + y_m = n - m, \qquad y_i \geq 0.$$

We know that the number of solutions $(y_1, \ldots, y_m)$ is the same as the number of ways to choose $n - m$ objects of m kinds, which is $C(n - m + m - 1, m - 1) = C(n - 1, m - 1)$.

Problem 3 (15 points): Use the binomial theorem to compute the coefficient a_2 of x^2 in the expansion

$$\left(x + \frac{2}{x}\right)^{10} = \sum_{k=-10}^{10} a_k x^k.$$

By the binomial theorem,

$$\left(x + \frac{2}{x}\right)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^{10-k} \left(\frac{2}{x}\right)^k = \sum_{k=0}^{10} \binom{10}{k} 2^k x^{10-2k}.$$

The exponent of x is 2 when $10 - 2k = 2$ or $k = 4$, so the coefficient of x^2 is

$$2^4 \binom{10}{4} = 16 \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 3360$$

Problem 4 (25 points): Define a sample space S by $S = \{1, 2, 3, 4, 5, 6\}$, and let the probability of any outcome x in S be $p(x) = 1/6$. Define random variables $f, g : S \to \mathbf{Z}$ by

$$f(x) = x \bmod 2, \qquad g(x) = x \bmod 3.$$

(a) Calculate $E(f)$ and $E(g)$.

Define sets F_i and G_j by $F_i = \{f = i\}$ for $i = 0, 1$ and $G_j = \{g = j\}$ for $j = 0, 1, 2$. Then $p(F_i) = 1/2$ and $p(G_j) = 1/3$, so

$$E(f) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$$

and

$$E(g) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 = 1.$$

(b) Calculate $V(f)$ and $V(g)$.

$$E(f^2) = \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

and

$$E(g^2) = \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 + \frac{1}{3} \cdot 2^2 = \frac{5}{3},$$

so $V(f) = 1/4$ and $V(g) = 2/3$.

(c) Prove that f and g are independent.

We need to show that $p(f = i, g = j) = p(f = i)p(g = j)$ or equivalently that $p(F_i \cap G_j) = p(F_i)p(G_j)$ for all i, j. Since $F_0 = \{2, 4, 6\}$, $F_1 = \{1, 3, 5\}$, $G_0 = \{3, 6\}$, $G_1 = \{1, 4\}$, $G_2 = \{2, 5\}$, we see that $|F_i \cap G_j| = 1$, so $p(F_i \cap G_j) = 1/6 = p(F_i)p(G_j)$ and f is independent of g.

(d) Calculate $E(f + g)$ and $V(f + g)$.

First, $E(f + g) = E(f) + E(g) = 1 + 1/2 = 3/2$. Second, since f and g are independent, their variances add: $V(f + g) = V(f) + V(g) = 11/12$.

Problem 5 (25 points): Consider the following pseudocode:

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function f(a: integer, b: nonnegative integer) if (b = 0)
f(a, b) := 1
else if (b mod 2 = 0)
f(a, b) := f(a, b / 2) * f(a, b / 2)
else
f(a, b) := a * f(a, b - 1)
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(a) Evaluate $\mathtt{f}(-3, 2)$.

Put $\mathtt{a} = -3$ and $\mathtt{b} = 2$. Since $\mathtt{b}$ is even and nonzero, $\mathtt{f}(\mathtt{a}, \mathtt{b}) = \mathtt{f}(\mathtt{a}, \mathtt{b}/2) * \mathtt{f}(\mathtt{a}, \mathtt{b}/2) = \mathtt{f}(-3, 1) * \mathtt{f}(-3, 1)$. Since 1 is odd and nonzero, $\mathtt{f}(-3, 1) = -3 * \mathtt{f}(-3, 1 - 1) = -3 * \mathtt{f}(-3, 0) = -3$. Thus $\mathtt{f}(-3, 2) = (-3) * (-3) = 9$.

(b) What function of $\mathtt{a}$ and $\mathtt{b}$ does this code calculate?

If $\mathtt{a} = \mathtt{b} = 0$, this code returns 1. Otherwise, it returns $\mathtt{a}^{\mathtt{b}}$. It computes $\mathtt{a}^{\mathtt{b}}$ recursively, using the fact that $\mathtt{a}^{\mathtt{b}} = (\mathtt{a}^{\mathtt{b}/2})^2$ if $\mathtt{b}$ is even and $\mathtt{a}^{\mathtt{b}} = \mathtt{a} \cdot \mathtt{a}^{\mathtt{b}-1}$ if $\mathtt{b}$ is odd.

(c) Use induction to prove that your answer to (b) is correct. (Hint: Start at $\mathtt{b = 1}$.)

We will use the second kind of induction, on $\mathtt{b}$, with $\mathtt{a}$ fixed. The case $\mathtt{b} = 0$ is clear, so we will start at $\mathtt{b} = 1$.

Base: $\mathtt{b} = 1$. Then $\mathtt{f}(\mathtt{a}, \mathtt{b}) = \mathtt{a} * \mathtt{f}(\mathtt{a}, 0) = \mathtt{a}$ is correct since $\mathtt{a}^1 = \mathtt{a}$.

Induction step: Consider two cases.

Case 1: $\mathtt{b}$ is even. Then by the induction hypothesis, $\mathtt{f}(\mathtt{a}, \mathtt{b}/2) = \mathtt{a}^{\mathtt{b}/2}$, so we have $\mathtt{f}(\mathtt{a}, \mathtt{b}) = (\mathtt{a}^{\mathtt{b}/2})^2 = \mathtt{a}^{\mathtt{b}}$.

Case 2: $\mathtt{b}$ is odd. Then by the induction hypothesis, $\mathtt{f}(\mathtt{a}, \mathtt{b} - 1) = \mathtt{a}^{\mathtt{b}-1}$, so we have $\mathtt{f}(\mathtt{a}, \mathtt{b}) = \mathtt{a} * \mathtt{a}^{\mathtt{b}-1} = \mathtt{a}^{\mathtt{b}}$.