

Prof. Bjorn Poonen
December 14, 2001

MATH 55 FINAL ANSWERS (white)

(1) (5 pts. each)
(a) TRUE

(b) FALSE

(c) FALSE

(d) FALSE

(e) TRUE

(2) (15 pts.) $a_1 = 25$, $a_2 = 626$, $a_n = 25a_{n-1} + a_{n-2}$ for $n \geq 3$
Alternatively: $a_0 = 1$, $a_1 = 25$, $a_n = 25a_{n-1} + a_{n-2}$ for $n \geq 2$

(3) (20 pts.) $a_n = (3n + 3)3^n + 2(-2)^n$
Alternatively: $a_n = (n + 1)3^{n+1} + 2(-2)^n$

(4) (15 pts.) $1/2$

(5) (10 pts. each)
(a) $21/2$
(b) $7/2$

(6) (10 pts.) $\{1, 3, 4, 6\}$
Also acceptable: 1346.

(7) (10 pts.) 13

(8) (15 pts.) 10

(9) (10 pts.) $\frac{1}{(1-x)(1-x^3)(1-x^5)}$

(10) (20 pts.) 5000

(11) (15 pts.) 40

(12) (10 pts.) 16

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MATH 55 FINAL SOLUTIONS (white)

Do not write your answers on this sheet. Instead please write your name, your student ID, your TA's name, your section time, "white," and all your answers in your blue books. **IMPORTANT:** Write your answers to problems 1–12 on the first page or two of your blue book, without the calculations you did to get the answers. To guarantee full credit and to qualify for partial credit, you should show your work on *later* pages of the blue book. Total: 200 pts., 170 minutes.

(1) (5 pts. each) For each of (a)–(e) below: If the statement is true (always), write *TRUE*. Otherwise write *FALSE*. (Please do not use the abbreviations *T* and *F*, since in handwriting they are sometimes indistinguishable.) No explanations are required in this problem.

(a) If a function $f(n)$ is $\Theta(2^n)$ as $n \rightarrow \infty$, then $f(n)$ is also $o(n!)$ as $n \rightarrow \infty$.

TRUE. First, $\lim_{n \rightarrow \infty} 2^n/n! = 0$, by the Ratio Test. Since $f(n)$ is $\Theta(2^n)$, there exists $c > 0$ such that $|f(n)| \leq c2^n$ for sufficiently large n . Thus, for sufficiently large n , the ratio $f(n)/n!$ is sandwiched between $-c2^n/n!$ and $c2^n/n!$. Both expressions on the right tend to 0 as $n \rightarrow \infty$, so $\lim_{n \rightarrow \infty} f(n)/n! = 0$. This means that $f(n) = o(n!)$, by definition.

(b) For any predicate $P(x)$, the propositions $\neg \forall x P(x)$ and $\exists x P(x)$ are logically equivalent.

FALSE. Here is a counterexample. Suppose that $P(x)$ is true for all x . Then $\neg \forall x P(x)$ is false, but $\exists x P(x)$ is true (provided that the universe of discourse is nonempty). So the propositions are not logically equivalent in this case.

(c) Exactly half of the 4-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ contain the number 5.

FALSE. There are $\binom{7}{4} = 35$ three-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$. Since 35 is odd, it cannot be that exactly 35/2 of them contain 5. In fact, the number of four-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ equals the number of three-element subsets of $\{1, 2, 3, 4, 6, 7\}$, which is $\binom{6}{3} = 20$.

(d) If A , B , and C are pairwise disjoint sets (possibly infinite), and $|A \cup B| = |A \cup C|$, then $|B| = |C|$.

FALSE. Let $A = \{2, 3, 4, \dots\}$, $B = \{0, 1\}$, and $C = \{1\}$. The function $f(n) = n + 1$ defines a bijection between $A \cup B$ and $A \cup C$, so $|A \cup B| = |A \cup C|$. But $|B| = 2 \neq 1 = |C|$. Thus we have a counterexample.

3

(e) The statement $p\{S\}q$ is true, where p is the assertion " $n = -1$ ", q is the assertion " $k = -1$ ", and S is the program segment consisting of the following six lines:

```

k := 1
while n ≠ 0
begin
    n := n - 1
    k := n * k
end

```

TRUE. The meaning of " $p\{S\}q$ " is that "if p is true at the beginning of the execution and the program terminates, then q is true at the end of the execution." Since this program does not terminate, the "if" part of this statement is false, so the whole statement is vacuously true.

(2) (15 pts.) For each integer $n \geq 1$, let a_n denote the number of strings of n letters in which every "q" is immediately followed by a "u". Find a recurrence relation and initial condition(s) that determine the sequence a_1, a_2, \dots . (Do not try to find an explicit formula for a_n .)

The number of such n -letter strings beginning with some letter other than "q" is $25a_{n-1}$, since these are constructed by choosing one of the 25 letters that is not "q", and following it by an $(n-1)$ -letter string in which every "q" is followed by "u". The number of such n -letter strings beginning with "q" is a_{n-2} , since these are constructed by following "qu" with an $(n-2)$ -letter string in which every "q" is followed by "u". By the Sum Rule, $a_n = 25a_{n-1} + a_{n-2}$. This argument makes sense for $n \geq 3$. Thus we need initial conditions to specify a_1 and a_2 .

We have $a_1 = 25$ (any letter but "q" is OK). Also, $a_2 = 25^2 + 1 = 626$ (the 25^2 two-letter strings with no q's are OK, but there is only 1 two-letter string with a "q" that is OK, namely "qu").

Alternatively, one could observe that $a_n = 25a_{n-1} + a_{n-2}$ holds also for $n = 2$ if we define $a_0 = 1$ (which counts the empty string). In this case, one would use the initial conditions $a_0 = 1$ and $a_1 = 25$ instead.

4

(3) (20 pts.) A sequence a_0, a_1, \dots satisfies $a_0 = 5$, $a_1 = 14$, and

$$a_n = a_{n-1} + 6a_{n-2} + 5(3^n)$$

for $n \geq 2$. Find an explicit formula for a_n .

The associated homogeneous recurrence is $a_n - a_{n-1} - 6a_{n-2} = 0$, which has characteristic polynomial $r^2 - r - 6 = 0$ and characteristic roots 3 and -2 . Thus the general solution to the homogeneous recurrence is $a_n = \alpha 3^n + \beta(-2)^n$, where α and β are arbitrary constants.

Since the 3 in the $5(3^n)$ is a characteristic root, we look for a particular solution to the original nonhomogeneous recurrence without initial conditions of the form $a_n = cn3^n$ for some constant c . In order for this to be a solution,

$$cn3^n = c(n-1)3^{n-1} + 6c(n-2)3^{n-2} + 5(3^n)$$

must hold for all n . This condition on c can be rewritten as

$$\begin{aligned} cn3^n &= (c/3)n3^n - (c/3)3^n + (6c/9)n3^n - (12c/9)3^n + 5(3^n), \\ 0 &= (-c/3 - 12c/9 + 5)3^n. \end{aligned}$$

This holds for all n if and only if $-c/3 - 12c/9 + 5 = 0$, that is, if and only if $c = 3$. Thus $a_n = 3n3^n$ is a particular solution to the original nonhomogeneous recurrence without initial conditions.

Hence the *general* solution to the original nonhomogeneous recurrence without initial conditions is

$$a_n = \alpha 3^n + \beta(-2)^n + 3n3^n.$$

Now we use the initial conditions to determine α and β :

$$\begin{aligned} 5 &= a_0 = \alpha + \beta \\ 14 &= a_1 = 3\alpha - 2\beta + 9. \end{aligned}$$

Solving this system for α and β yields $\alpha = 3$, $\beta = 2$, so the final answer is

$$a_n = 3(3^n) + 2(-2)^n + 3n3^n = (3n + 3)3^n + 2(-2)^n.$$

(4) (15 pts.) A fair coin is flipped. If it comes up heads, then two more fair coins are flipped; if instead the first coin comes up tails, then only one more fair coin is flipped. (Thus the total number of coin flips is either 3 or 2.) What is the probability that there is exactly one tail in all the flips?

The sample space is $\{HHH, HHT, HTH, HTT, TH, TT\}$. The total probability of the first four outcomes equals the probability that the first coin comes up heads, which is $1/2$. Also each of the first four outcomes is equally likely, so $P(HHH) = P(HHT) = P(HTH) = P(HTT) = 1/8$. Similarly $P(\{TH, TT\}) = 1/2$, and $P(TH) = P(TT) = 1/4$. The event A that there is exactly one tail is $\{HHT, HTH, TH\}$, so $P(A) = P(HHT) + P(HTH) + P(TH) = 1/8 + 1/8 + 1/4 = 1/2$.

5

(5) (10 pts. each) A game is played by rolling a fair 6-sided die 21 times. The player scores two points for each 6 that is rolled and one point for each 5 that is rolled.

(a) What is the expected value of the total score?

(b) What is the standard deviation of the total score?

Let f_i be the number of points scored with the i^{th} roll, so that the total score is $f_1 + f_2 + \cdots + f_{21}$. Then

$$E(f_i) = \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}1 + \frac{1}{6}2 = \frac{1}{2}$$

$$E(f_i^2) = \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}1 + \frac{1}{6}4 = \frac{5}{6}$$

$$V(f_i) = E(f_i^2) - E(f_i)^2 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12},$$

$$E(f) = E(f_1) + \cdots + E(f_{21}) = 21(1/2) = 21/2,$$

$$V(f) = V(f_1) + \cdots + V(f_{21}) = 21(7/12) = 49/4.$$

Thus the expected value of f is $21/2$, and the standard deviation is $\sqrt{49/4} = 7/2$.

(6) (10 pts.) When the 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ are listed in lexicographic order, which 4-combination is eighth on the list?

The 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ in lexicographic order are (omitting set braces)

1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 3456

so the answer is $\{1, 3, 4, 6\}$.

(7) (10 pts.) How many cards must be taken from a standard deck to guarantee that there will exist some suit such that the selected cards include at least four cards of that suit? (In a standard deck, there are four suits, and 13 cards of each suit, for a total of 52 cards.)

The answer is 13. Taking 12 cards is not enough, because then it could happen that there are only 3 cards in each suit. With 13 cards, the Generalized Pigeonhole Principle implies that some suit contains at least $\lceil 13/4 \rceil = 4$ of the selected cards.

(8) (15 pts.) Alice stands at the bottom of a staircase. How many ways can she take a sequence of 5 steps, where each step must be either a step up or a step down? (Be careful: if at any time she is at the bottom of the staircase, her next step must be upwards, obviously! Assume that the staircase has more than 5 steps.)

Represent the sequences by strings of U 's and D 's. The first step must be U . We categorize the sequences according to the direction of the next step.

First case: the initial segment UU can be completed in $2^3 - 1 = 7$ ways, since any three steps other than DDD can follow UU .

Second case: UD can be completed in $2^2 - 1 = 3$ ways, since the third step must be U , and then anything is possible for the last two steps except DD .

By the Sum Rule, the total number of possibilities is $7 + 3 = 10$.

(This problem could also be solved with a tree diagram.)

6

(9) (10 pts.) For each nonnegative integer n , let a_n be the number of solutions (x, y, z) to the equation $x + 3y + 5z = n$ in nonnegative integers. Express the generating function of the sequence a_0, a_1, a_2, \dots in closed form.

The number a_n is the coefficient of x^n in

$$(1+x+x^2+x^3+\dots)(1+x^3+x^6+x^9+\dots)(1+x^5+x^{10}+x^{15}+\dots) = \frac{1}{(1-x)(1-x^3)(1-x^5)},$$

so this power series is the generating function.

(10) (20 pts.) How many functions $f: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ have the property that $|f^{-1}(\{1, 2\})| = 2$? (Note: $f^{-1}(\{1, 2\})$ denotes the inverse image of the set $\{1, 2\}$ under f .)

The condition on f says that exactly two of the elements of $\{1, 2, 3, 4, 5\}$ should map into the subset $\{1, 2\}$ of the range. (The other three elements of $\{1, 2, 3, 4, 5\}$ must then map into $\{3, 4, 5, 6, 7\}$.) The selection of such an f can be broken down into three steps:

Step 1: Choose the two elements a, b of $\{1, 2, 3, 4, 5\}$ that are to map into $\{1, 2\}$.

Step 2: Choose a value for $f(a)$ (either 1 or 2), and a value for $f(b)$.

Step 3: For each of the three elements of $\{1, 2, 3, 4, 5\}$ not equal to a or b , decide which element of $\{3, 4, 5, 6, 7\}$ to map it to.

There are $\binom{5}{2}$ ways to complete Step 1. For each way to complete Step 1, there are then 2^2 ways to complete Step 2, and 5^3 ways to complete Step 3. By the Product Rule, the answer is

$$\binom{5}{2} 2^2 5^3 = 10 \cdot 4 \cdot 125 = 5000.$$

(11) (15 pts.) How many positive integers less than or equal to 150 are relatively prime to 30?

Let $U = \{1, 2, \dots, 150\}$. Say that an integer n in U has property P_2 if $2|n$. Similarly define P_3 and P_5 . Since the prime factorization of 30 is $2 \cdot 3 \cdot 5$, an integer n is relatively prime to 30 if and only if it has none of the properties P_2, P_3, P_5 . By the Principle of Inclusion-Exclusion, the answer to the problem is

$$N - N(P_2) - N(P_3) - N(P_5) + N(P_2P_3) + N(P_2P_5) + N(P_3P_5) - N(P_2P_3P_5)$$

where $N = |U| = 150$, $N(P_2) = \lfloor 150/2 \rfloor$ denotes the number of integers in U having property P_2 , $N(P_2P_5) = \lfloor 150/10 \rfloor$ denotes the number of integers in U having properties P_2 and P_5 (that is, those divisible by 10), and so on. Thus the answer is

$$\begin{aligned} & 150 - \lfloor 150/2 \rfloor - \lfloor 150/3 \rfloor - \lfloor 150/5 \rfloor + \lfloor 150/6 \rfloor + \lfloor 150/10 \rfloor + \lfloor 150/15 \rfloor - \lfloor 150/30 \rfloor \\ & = 150 - 75 - 50 - 30 + 25 + 15 + 10 - 5 \\ & = 40. \end{aligned}$$

7

(12) (10 pts.) How many times will the command

wastetime(6)

print "Hi" if the procedure wastetime is defined by the pseudocode below?

```

procedure wastetime(n: positive integer)
  if n = 1 then
    print "Hi"
  else for i := 1 to n - 1
    wastetime(i)

```

Let a_n denote the number of times that the command $wastetime(n)$ prints "Hi". Then $a_1 = 1$, and $a_n = a_1 + a_2 + \cdots + a_{n-1}$. Calculating one term at a time, we find

$$\begin{aligned}
 a_2 &= 1 \\
 a_3 &= 1 + 1 = 2 \\
 a_4 &= 1 + 1 + 2 = 4 \\
 a_5 &= 1 + 1 + 2 + 4 = 8 \\
 a_6 &= 1 + 1 + 2 + 4 + 8 = 16
 \end{aligned}$$

Thus the answer is 16.

(One could also use induction to prove that $a_n = 2^{n-2}$ for all $n \geq 2$.)

(13) (15 pts.) Prove the identity

$$\binom{0}{6} + \binom{1}{6} + \binom{2}{6} + \cdots + \binom{n-1}{6} = \binom{n}{7}$$

for all integers $n \geq 1$. (Suggestion: use induction.)

For each $n \geq 1$, let P_n denote the proposition

$$\binom{0}{6} + \binom{1}{6} + \binom{2}{6} + \cdots + \binom{n-1}{6} = \binom{n}{7}.$$

We will prove P_n for all $n \geq 1$ by induction on n .

Base case: P_1 asserts that $\binom{0}{6} = \binom{1}{7}$. This is true, since both sides are 0.

Inductive step: Suppose that $n \geq 1$ and P_n holds. Then

$$\begin{aligned}
 \binom{0}{6} + \cdots + \binom{n-1}{6} + \binom{n}{6} &= \binom{n}{7} + \binom{n}{6} \quad (\text{inductive hypothesis}) \\
 &= \binom{n+1}{7} \quad (\text{Pascal's identity})
 \end{aligned}$$

so P_{n+1} holds. This completes the proof.

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MATH 55 FINAL ANSWERS (yellow)

(1) (5 pts. each)
(a) FALSE

(b) FALSE

(c) TRUE

(d) FALSE

(e) TRUE

(2) (10 pts.) $\{1, 3, 4, 5\}$
Also acceptable: 1345.

(3) (10 pts.) 9

(4) (15 pts.) 10

(5) (10 pts.) $\frac{1}{(1-x)(1-x^2)(1-x^4)}$

(6) (20 pts.) 2000

(7) (15 pts.) 80

(8) (10 pts.) 32

(9) (15 pts.) $a_1 = 25, a_2 = 626, a_n = 25a_{n-1} + a_{n-2}$ for $n \geq 3$.
Alternatively: $a_0 = 1, a_1 = 25, a_n = 25a_{n-1} + a_{n-2}$ for $n \geq 2$.

(10) (20 pts.) $a_n = (3n+3)3^n + (-2)^n$
Alternatively: $a_n = (n+1)3^{n+1} + (-2)^n$

(11) (15 pts.) $3/8$

(12) (10 pts. each)
(a) $21/2$
(b) $7/2$

2

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(a) Exactly half of the 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ contain the number 5.

FALSE. There are $\binom{7}{3} = 35$ three-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$. Since 35 is odd, it cannot be that exactly $35/2$ of them contain 5. In fact, the number of three-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ equals the number of two-element subsets of $\{1, 2, 3, 4, 6, 7\}$, which is $\binom{6}{2} = 15$.

(b) If A , B , and C are pairwise disjoint sets (possibly infinite), and $|A \cup B| = |A \cup C|$, then $|B| = |C|$.

FALSE. Let $A = \{2, 3, 4, \dots\}$, $B = \{0, 1\}$, and $C = \{1\}$. The function $f(n) = n + 1$ defines a bijection between $A \cup B$ and $A \cup C$, so $|A \cup B| = |A \cup C|$. But $|B| = 2 \neq 1 = |C|$. Thus we have a counterexample.

(c) If a function $f(n)$ is $O(2^n)$ as $n \rightarrow \infty$, then $f(n)$ is also $o(n!)$ as $n \rightarrow \infty$.

TRUE. First, $\lim_{n \rightarrow \infty} 2^n/n! = 0$, by the Ratio Test. Since $f(n)$ is $O(2^n)$, there exists $c > 0$ such that $|f(n)| \leq c2^n$ for sufficiently large n . Thus, for sufficiently large n , the ratio $f(n)/n!$ is sandwiched between $-c2^n/n!$ and $c2^n/n!$. Both expressions on the right tend to 0 as $n \rightarrow \infty$, so $\lim_{n \rightarrow \infty} f(n)/n! = 0$. This means that $f(n) = o(n!)$, by definition.

(d) For any predicate $P(x)$, the propositions $\neg \exists x P(x)$ and $\forall x P(x)$ are logically equivalent.

FALSE. Here is a counterexample. Suppose that $P(x)$ is true for all x . Then $\neg \exists x P(x)$ is false (provided that the universe of discourse is nonempty), but $\forall x P(x)$ is true. So the propositions are not logically equivalent in this case.

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(e) The statement $p\{S\}q$ is true, where p is the assertion " $n = -1$ ", q is the assertion " $k = -1$ ", and S is the program segment consisting of the following six lines:

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k := 1
while n ≠ 0
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    k := n * k
end

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TRUE. The meaning of " $p\{S\}q$ " is that "if p is true at the beginning of the execution and the program terminates, then q is true at the end of the execution." Since this program does not terminate, the "if" part of this statement is false, so the whole statement is vacuously true.

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The 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ in lexicographic order are (omitting set braces)

1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 3456

so the answer is $\{1, 3, 4, 5\}$.

(3) (10 pts.) How many cards must be taken from a standard deck to guarantee that there will exist some suit such that the selected cards include at least three cards of that suit? (In a standard deck, there are four suits, and 13 cards of each suit, for a total of 52 cards.)

The answer is 9. Taking 8 cards is not enough, because then it could happen that there are only 2 cards in each suit. With 9 cards, the Generalized Pigeonhole Principle implies that some suit contains at least $\lceil 9/4 \rceil = 3$ of the selected cards.

(4) (15 pts.) Alice stands at the bottom of a staircase. How many ways can she take a sequence of 5 steps, where each step must be either a step up or a step down? (Be careful: if at any time she is at the bottom of the staircase, her next step must be upwards, obviously! Assume that the staircase has more than 5 steps.)

Represent the sequences by strings of U 's and D 's. The first step must be U . We categorize the sequences according to the direction of the next step.

First case: the initial segment UU can be completed in $2^3 - 1 = 7$ ways, since any three steps other than DDD can follow UU .

Second case: UD can be completed in $2^2 - 1 = 3$ ways, since the third step must be U , and then anything is possible for the last two steps except DD .

By the Sum Rule, the total number of possibilities is $7 + 3 = 10$.

(This problem could also be solved with a tree diagram.)

(5) (10 pts.) For each nonnegative integer n , let a_n be the number of solutions (x, y, z) to the equation $x + 2y + 4z = n$ in nonnegative integers. Express the generating function of the sequence a_0, a_1, a_2, \dots in closed form.

The number a_n is the coefficient of x^n in

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)(1+x^4+x^8+x^{12}+\dots) = \frac{1}{(1-x)(1-x^2)(1-x^4)},$$

so this power series is the generating function.

4

(6) (20 pts.) How many functions $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ have the property that $|f^{-1}(\{1, 2\})| = 3$? (Note: $f^{-1}(\{1, 2\})$ denotes the inverse image of the set $\{1, 2\}$ under f .)

The condition on f says that exactly three of the elements of $\{1, 2, 3, 4, 5\}$ should map into the subset $\{1, 2\}$ of the range. (The other two elements of $\{1, 2, 3, 4, 5\}$ must then map into $\{3, 4, 5, 6, 7\}$.) The selection of such an f can be broken down into three steps:

Step 1: Choose the three elements a, b, c of $\{1, 2, 3, 4, 5\}$ that are to map into $\{1, 2\}$.

Step 2: Choose a value for $f(a)$ (either 1 or 2), and a value for $f(b)$, and a value for $f(c)$.

Step 3: For each of the two elements of $\{1, 2, 3, 4, 5\}$ not equal to a, b , or c , decide which element of $\{3, 4, 5, 6, 7\}$ to map it to.

There are $\binom{5}{3}$ ways to complete Step 1. For each way to complete Step 1, there are then 2^3 ways to complete Step 2, and 5^2 ways to complete Step 3. By the Product Rule, the answer is

$$\binom{5}{3} 2^3 5^2 = 10 \cdot 8 \cdot 25 = 2000.$$

(7) (15 pts.) How many positive integers less than or equal to 300 are relatively prime to 30?

Let $U = \{1, 2, \dots, 300\}$. Say that an integer n in U has property P_2 if $2|n$. Similarly define P_3 and P_5 . Since the prime factorization of 30 is $2 \cdot 3 \cdot 5$, an integer n is relatively prime to 30 if and only if it has none of the properties P_2, P_3, P_5 . By the Principle of Inclusion-Exclusion, the answer to the problem is

$$N - N(P_2) - N(P_3) - N(P_5) + N(P_2P_3) + N(P_2P_5) + N(P_3P_5) - N(P_2P_3P_5)$$

where $N = |U| = 300$, $N(P_2) = \lfloor 300/2 \rfloor$ denotes the number of integers in U having property P_2 , $N(P_2P_5) = \lfloor 300/10 \rfloor$ denotes the number of integers in U having properties P_2 and P_5 (that is, those divisible by 10), and so on. Thus the answer is

$$\begin{aligned} & 300 - \lfloor 300/2 \rfloor - \lfloor 300/3 \rfloor - \lfloor 300/5 \rfloor + \lfloor 300/6 \rfloor + \lfloor 300/10 \rfloor + \lfloor 300/15 \rfloor - \lfloor 300/30 \rfloor \\ &= 300 - 150 - 100 - 60 + 50 + 30 + 20 - 10 \\ &= 80. \end{aligned}$$

5

(8) (10 pts.) How many times will the command

wastetime(7)

print "Hi" if the procedure wastetime is defined by the pseudocode below?

```

procedure wastetime(n: positive integer)
  if n = 1 then
    print "Hi"
  else for i := 1 to n - 1
    wastetime(i)

```

Let a_n denote the number of times that the command $wastetime(n)$ prints "Hi". Then $a_1 = 1$, and $a_n = a_1 + a_2 + \dots + a_{n-1}$. Calculating one term at a time, we find

$$\begin{aligned}
 a_2 &= 1 \\
 a_3 &= 1 + 1 = 2 \\
 a_4 &= 1 + 1 + 2 = 4 \\
 a_5 &= 1 + 1 + 2 + 4 = 8 \\
 a_6 &= 1 + 1 + 2 + 4 + 8 = 16 \\
 a_7 &= 1 + 1 + 2 + 4 + 8 + 16 = 32.
 \end{aligned}$$

Thus the answer is 32.

(One could also use induction to prove that $a_n = 2^{n-2}$ for all $n \geq 2$.)

(9) (15 pts.) For each integer $n \geq 1$, let a_n denote the number of strings of n letters in which every "q" is immediately followed by a "u". Find a recurrence relation and initial condition(s) that determine the sequence a_1, a_2, \dots . (Do not try to find an explicit formula for a_n .)

The number of such n -letter strings beginning with some letter other than "q" is $25a_{n-1}$, since these are constructed by choosing one of the 25 letters that is not "q", and following it by an $(n-1)$ -letter string in which every "q" is followed by "u". The number of such n -letter strings beginning with "q" is a_{n-2} , since these are constructed by following "qu" with an $(n-2)$ -letter string in which every "q" is followed by "u". By the Sum Rule, $a_n = 25a_{n-1} + a_{n-2}$. This argument makes sense for $n \geq 3$. Thus we need initial conditions to specify a_1 and a_2 .

We have $a_1 = 25$ (any letter but "q" is OK). Also, $a_2 = 25^2 + 1 = 626$ (the 25^2 two-letter strings with no q's are OK, but there is only 1 two-letter string with a "q" that is OK, namely "qu").

Alternatively, one could observe that $a_n = 25a_{n-1} + a_{n-2}$ holds also for $n = 2$ if we define $a_0 = 1$ (which counts the empty string). In this case, one would use the initial conditions $a_0 = 1$ and $a_1 = 25$ instead.

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(10) (20 pts.) A sequence a_0, a_1, \dots satisfies $a_0 = 4$, $a_1 = 16$, and

$$a_n = a_{n-1} + 6a_{n-2} + 5(3^n)$$

for $n \geq 2$. Find an explicit formula for a_n .

The associated homogeneous recurrence is $a_n - a_{n-1} - 6a_{n-2} = 0$, which has characteristic polynomial $r^2 - r - 6 = 0$ and characteristic roots 3 and -2 . Thus the general solution to the homogeneous recurrence is $a_n = \alpha 3^n + \beta(-2)^n$, where α and β are arbitrary constants.

Since the 3 in the $5(3^n)$ is a characteristic root, we look for a particular solution to the original nonhomogeneous recurrence without initial conditions of the form $a_n = cn3^n$ for some constant c . In order for this to be a solution,

$$cn3^n = c(n-1)3^{n-1} + 6c(n-2)3^{n-2} + 5(3^n)$$

must hold for all n . This condition on c can be rewritten as

$$\begin{aligned} cn3^n &= (c/3)n3^n - (c/3)3^n + (6c/9)n3^n - (12c/9)3^n + 5(3^n), \\ 0 &= (-c/3 - 12c/9 + 5)3^n. \end{aligned}$$

This holds for all n if and only if $-c/3 - 12c/9 + 5 = 0$, that is, if and only if $c = 3$. Thus $a_n = 3n3^n$ is a particular solution to the original nonhomogeneous recurrence without initial conditions.

Hence the *general* solution to the original nonhomogeneous recurrence without initial conditions is

$$a_n = \alpha 3^n + \beta(-2)^n + 3n3^n.$$

Now we use the initial conditions to determine α and β :

$$\begin{aligned} 4 &= a_0 = \alpha + \beta \\ 16 &= a_1 = 3\alpha - 2\beta + 9. \end{aligned}$$

Solving this system for α and β yields $\alpha = 3$, $\beta = 1$, so the final answer is

$$a_n = 3(3^n) + (-2)^n + 3n3^n = (3n + 3)3^n + (-2)^n.$$

(11) (15 pts.) A fair coin is flipped. If it comes up heads, then two more fair coins are flipped; if instead the first coin comes up tails, then only one more fair coin is flipped. (Thus the total number of coin flips is either 3 or 2.) What is the probability that there is exactly one head in all the flips?

The sample space is $\{HHH, HHT, HTH, HTT, TH, TT\}$. The total probability of the first four outcomes equals the probability that the first coin comes up heads, which is $1/2$. Also each of the first four outcomes is equally likely, so $P(HHH) = P(HHT) = P(HTH) = P(HTT) = 1/8$. Similarly $P(\{TH, TT\}) = 1/2$, and $P(TH) = P(TT) = 1/4$. The event A that there is exactly one head is $\{HTT, TH\}$, so $P(A) = P(HTT) + P(TH) = 1/8 + 1/4 = 3/8$.

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(12) (10 pts. each) A game is played by rolling a fair 6-sided die 21 times. The player scores two points for each 6 that is rolled and one point for each 5 that is rolled.

(a) What is the expected value of the total score?

(b) What is the standard deviation of the total score?

Let f_i be the number of points scored with the i^{th} roll, so that the total score is $f_1 + f_2 + \cdots + f_{21}$. Then

$$E(f_i) = \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}1 + \frac{1}{6}2 = \frac{1}{2}$$

$$E(f_i^2) = \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}0 + \frac{1}{6}1 + \frac{1}{6}4 = \frac{5}{6}$$

$$V(f_i) = E(f_i^2) - E(f_i)^2 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$$

$$E(f) = E(f_1) + \cdots + E(f_{21}) = 21(1/2) = 21/2,$$

$$V(f) = V(f_1) + \cdots + V(f_{21}) = 21(7/12) = 49/4.$$

Thus the expected value of f is $21/2$, and the standard deviation is $\sqrt{49/4} = 7/2$.

(13) (15 pts.) Prove the identity

$$\binom{0}{7} + \binom{1}{7} + \binom{2}{7} + \cdots + \binom{n-1}{7} = \binom{n}{8}$$

for all integers $n \geq 1$. (Suggestion: use induction.)

For each $n \geq 1$, let P_n denote the proposition

$$\binom{0}{7} + \binom{1}{7} + \binom{2}{7} + \cdots + \binom{n-1}{7} = \binom{n}{8}.$$

We will prove P_n for all $n \geq 1$ by induction on n .

Base case: P_1 asserts that $\binom{0}{7} = \binom{1}{8}$. This is true, since both sides are 0.

Inductive step: Suppose that $n \geq 1$ and P_n holds. Then

$$\begin{aligned} \binom{0}{7} + \cdots + \binom{n-1}{7} + \binom{n}{7} &= \binom{n}{8} + \binom{n}{7} \quad (\text{inductive hypothesis}) \\ &= \binom{n+1}{8} \quad (\text{Pascal's identity}) \end{aligned}$$

so P_{n+1} holds. This completes the proof.