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MATH 53 1st MIDTERM

Write your name and section number on EVERY page of this exam. Use only the page containing a given problem for the answer to that problem (both sides if needed). At the end, separate the pages and put them in appropriate piles. Thanks!

Good Luck!

Name: ___ ______________

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Problem 1. Suppose that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = 6$. What is the length of $\mathbf{a} \times \mathbf{b}$?

$$|a \times b| = |a||b| \sin\theta$$

$$a \cdot b = |a||b| \cos\theta$$

$$6 = 4 \cdot 3 \cos\theta$$

$$\tfrac{1}{2} = \cos\theta$$

$$\theta = \tfrac{\pi}{3}$$

$$|a \times b| = 12 \sin(\tfrac{\pi}{3})$$

$$= \cancel{12}^{6} \left(\frac{\sqrt{3}}{\cancel{2}}\right) = 6\sqrt{3}$$

$$\frac{20}{20}$$

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Problem 2. Show that the quadric

$$\frac{x^2}{8} + y^2 - \frac{z^2}{9} = 1$$

contains the curve

$$x = 2\cosh t\,,\ y = \frac{1}{\sqrt{2}}\cosh t\,,\ z = 3\sinh t\,,$$

and sketch the quadric (do not sketch the curve!). What is the name of this quadric?

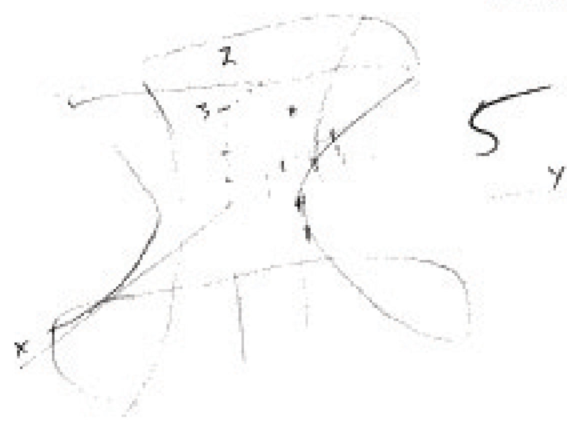
$(\cosh^2 t - \sinh^2 t = 1)$

$\frac{x}{2} = \cosh t \qquad \sqrt{2}\,y = \cosh t \qquad \frac{z}{3} = \sinh t$ 5

$\left.\frac{x^2}{8} + y^2 - \frac{z^2}{9} = 1\right\} \Rightarrow \frac{(2\cosh t)^2}{8} + \left(\frac{1}{\sqrt{2}}\cosh t\right)^2 - \frac{(3\sinh t)^2}{9} = 1$

$\frac{4\cosh^2 t}{8} + \frac{\cosh^2 t}{2} - \frac{9\sinh^2 t}{9} = 1$

$\cosh^2 t - \sinh^2 t = 1$ 5



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if $x = 0$ $\quad y^2 - \frac{z^2}{9} = 1$

hyperbola ~~$3y^2 - z^2$~~

$y = 0$ $\quad \frac{x^2}{8} - \frac{z^2}{9} = 1$

hyperbola

1 sheeted hyperboloid

along z axis 5

$z = 0$ $\quad \frac{x^2}{8} + y^2 = 1$

ellipse

$a = 2\sqrt{2} \quad b = 1$

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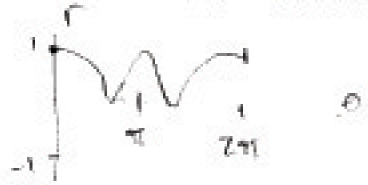
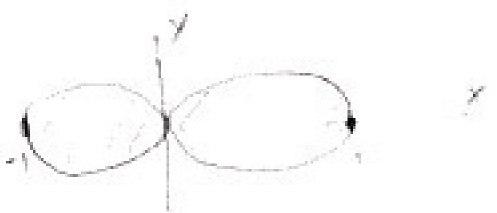
Problem 3. Sketch the curve $r = \cos^2\theta$ and find the area enclosed by it.



$r \geq 0$



No slope -2

$\underset{[0,2\pi]}{\text{area}} = 4 \times \underset{[0,\frac{\pi}{2}]}{\text{area}}$

$A = \frac{1}{2}\int_a^b r^2\, d\theta$

$= 4\frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos^2\theta)^2\, d\theta$

$\cos^4\theta = (\cos^2\theta)^2$

$= 2\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$

$\cos^2 = \frac{1}{2} + \frac{\cos 2\theta}{2}$

$= 2\int_0^{\frac{\pi}{2}}\left(\frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{1}{8} + \frac{\cos 4\theta}{8}\right)d\theta$

$\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)^2$

$= \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$

$= \frac{1}{4} + \frac{\cos 2\theta}{2} + \left(\frac{1 + \cos 4\theta}{2 \cdot 4}\right)$

$= 2\left(\frac{3}{8}\theta + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right)\Big|_0^{\frac{\pi}{2}}$

$= 2\left(\frac{3\pi}{16} + 0 + 0 - 0\right)$

$= \frac{3\pi}{8}$

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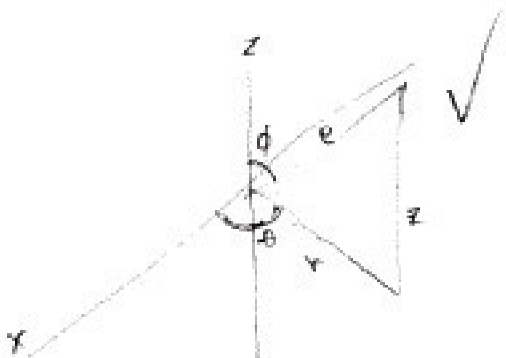
Problem 4. State the relations between the rectangular coordinates (x, y, z) and the spherical coordinates (ρ, θ, ϕ) (accompany this by a figure). Sketch the solid described by $\rho \leq 1$ and $\phi \geq \pi/3$.



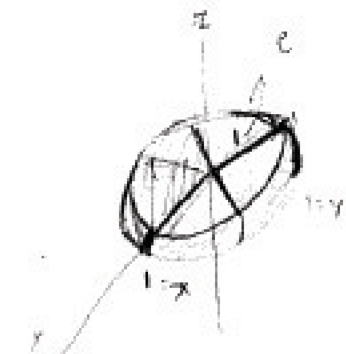
$y:\ \rho \sin\phi \cos\theta$

$y:\ \rho \sin\phi \sin\theta$

$z:\ \rho \cos\phi$

$r = \rho \sin\phi$

$\rho \leq 1$ $\quad \phi \geq \frac{\pi}{3}$ down to $z=0$ plane



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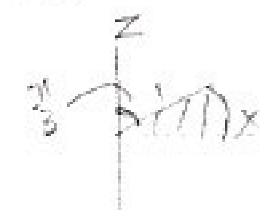
"circular wedge of cheese w/ center to edge angle of $\frac{\pi}{3}$"



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Problem 5. Show that the function $u(x,y) = y^{-\frac{1}{2}}e^{-x^2/(4y)}$ solves the following partial differential equation:

$$u_y = u_{xx}.$$

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$$u(x,y) = \frac{1}{y^{\frac{1}{2}}e^{\frac{x^2}{4y}}}$$

$$u_y = \underbrace{-\frac{1}{2}y^{-\frac{3}{2}}e^{-\frac{x^2}{4y}} + y^{-\frac{1}{2}}e^{-\frac{x^2}{4y}}\left(\frac{x^2}{4y^2}\right)}$$

$$\frac{x^2y^{-\frac{5}{2}}e^{-\frac{x^2}{4y}}}{4}$$

$$u_x = y^{-\frac{1}{2}}e^{-\frac{x^2}{4y}}\left(\frac{-2x}{4y}\right) = \frac{-2xe^{-\frac{x^2}{4y}}}{4y^{3/2}}$$

$$u_{xx} = \frac{-2e^{-\frac{x^2}{4y}} + (-2x)e^{-\frac{x^2}{4y}}\left(-\frac{2x}{4y}\right)}{4y^{\frac{3}{2}}}$$

$$u_{xx} = \frac{-2e^{-\frac{x^2}{4y}}}{4y^{\frac{3}{2}}} + \frac{4x^2e^{-\frac{x^2}{4y}}}{4\cdot 16y^{\frac{5}{2}}}$$

$$u_{xx} = -\frac{1}{2}y^{-\frac{3}{2}}e^{-\frac{x^2}{4y}} + \frac{x^2y^{-\frac{5}{2}}e^{-\frac{x^2}{4y}}}{4}$$

$u_y = u_{xx}$