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MATH 53 2nd MIDTERM

Write your name and section number on EVERY page of this exam. Use only the page containing a given problem for the answer to that problem (both sides if needed). At the end, please give the exam to your TA

Good Luck!

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Problem 1 (25 pts).

a) Evaluate $\int\int_D e^x dA$ where D is the triangle with vertices at $(0,0)$, $(2,4)$, and $(6,0)$. 13

b) Evaluate $\int_0^1 \int_{x^2}^1 x^3 \sin y^3 dy dx$. 11

a.

$\int_0^4 \int_{y/2}^{2} e^x\,dx\,dy + \int_0^4 \int_2^{6-y} e^x\,dx\,dy$

$= \int_0^4 e^x\Big]_{y/2}^{2} dy + \int_0^4 e^x\Big]_2^{6-y} dy$

$= \int_0^4 (e^2 - e^{\frac{1}{2}y})\,dy + \int_0^4 e^{6-y} - e^2\,dy$

$= ye^2 - 2e^{\frac{1}{2}y}\Big]_0^4 + (-e^{6-y} - ye^2)\Big]_0^4$

$= 4e^2 - 2e^2 - (0-2) + (-e^2 - 4e^2 - (-e^6 - 0))$

$= 2e^2 + 2 - e^2 - 4e^2 + e^6 = e^6 - 3e^2 + 2$

b.

$\int_0^1 \int_0^{\sqrt{y}} x^3 \sin y^3\,dx\,dy = \int_0^1 \frac{x^4}{4} \sin y^3\Big]_{x=0}^{x=\sqrt{y}}$

$= \int_0^1 \frac{1}{4} y^2 \sin y^3)\,dy = -\frac{1}{3}\cos(y^3)\Big]_0^1 = -\frac{1}{3}\cos(1) - (-\frac{1}{3})$

$= \frac{1}{3} - \frac{1}{3}\cos(1)$

$u = y^3$

$du = 3y^2\,dy$

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Problem 2 (15 pts). Define the Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

and *state carefully* the Change of Variables Theorem in dimension three.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

cross-product

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} + \frac{\partial x}{\partial w}\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} -$$

$$\left(\frac{\partial x}{\partial w}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial w}\frac{\partial z}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\frac{\partial z}{\partial w}\right)$$

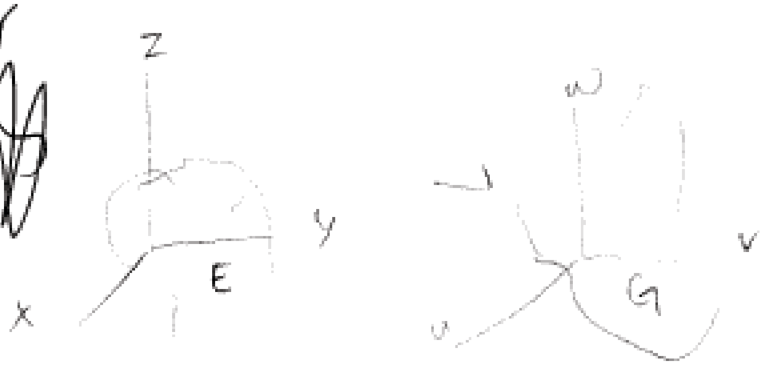
$$\iiint_{E} f(x,y,z)\, dz\, dy\, dx$$

E in xyz coordinates

$$\iiint_{G} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$

G in u,v,w coordinates

E & G are 3-dimensional ~~domain~~ volumes which is the volume of integration



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Problem 3. (15 pts) Evaluate the Jacobian of the transformation

$$x = u + v^2 + w, \quad y = v + w^4, \quad z = u - w.$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2v & 1 \\ 0 & 1 & 4w^3 \\ -1 & 0 & -1 \end{vmatrix}$$

$$= [-1 + 8vw^3 + 0] - [1 + 0 + 0]$$

$$= |-2 + 8vw^3| \leftarrow \text{no absolute value here}$$

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Problem 4 (25 pts). a) State carefully the second derivative test for determining a *local maximum*.

b) Consider the function

$$f(x, y) = x^2 + \frac{1}{3}x^3 - xy^2 + y^2 .$$

Find the absolute minimum and maximum of $f(x, y)$ inside the unit disc $x^2+y^2 \leq 1$.

a. $D > 0$ $\left(D: f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2 \right)$

$f_{xx} < 0$ $\quad \underline{f_x = f_y = 0}$

critical pts: $(0,0)$, $(1,0)$ on border

b. $f_x : 2x + x^2 - y^2 : 0$ $\quad y^2 + 2y = y^2$

$f_y : -2xy + 2y : 0$ $\quad 2y = 2xy$

$1 = x$

$0 = y$

$f(0,0) = 0$

(15)

boundary $\quad x^2 + y^2 = 1$

$y = \pm\sqrt{1 - y^2}$

$x = x$

$-1 \leq x \leq 1$

$f(x,y) = x^2 + \frac{1}{3}x^3 - x(1-x^2) + (1-x^2)$

$f(x,y) = x^2 + \frac{1}{3}x^3 - x + x^3 + 1 - x^2$

$f(x,y) = \frac{4}{3}x^3 - x + 1$ ✓

wrong sign / for x $\quad f_x(x,y) = 4x^2 - 1$

$f_x(x,y) = 4(x-1)(x+1)$

$x = \pm\frac{1}{2}$

min: $x = -1$, $y = 0$ $\quad f(x,y) = \frac{4}{3} + 1 + 1 = \frac{2}{3}$

max: $x = 1$, $y = 0$ $\quad f(x,y) = \frac{4}{3}$

min: $f(0,0) = 0$

max: $f(1,0) = \frac{4}{3}$

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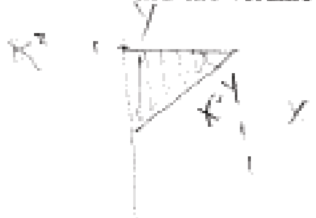
Problem 5 (20 pts) Consider the following solid in $\mathbb{R}^3$:

$$R = \{(x,y,z) \mid 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le x + y^2\}.$$

Find the volume of R and the area the surface which forms its boundary.



$$V = \int_0^1 \int_x^1 \int_0^{x+y^2} (1)\, dz\, dy\, dx$$

$$= \int_0^1 \int_x^1 z \Big]_0^{x+y^2} dy\, dx = \int_0^1 \int_x^1 x + y^2\, dy\, dx$$

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$$\int_0^1 xy + \frac{y^3}{3} \Big]_x^1 dx = \int_0^1 x + \frac{1}{3} - \left(x^2 + \frac{x^3}{3}\right) dx$$

$$= \frac{x^2}{2} + \frac{1}{3}x - \frac{x^3}{3} - \frac{x^4}{12} \Big]_0^1 = \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{12}$$

$$= \frac{6-1}{12} = \frac{5}{12}$$

$$SA: \int_0^1 \int_x^1 \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy\, dx = \int_0^1 \int_x^1 \sqrt{1 + 1^2 + (2y)^2}\, dy\, dx$$

$$z = f(x,y) = x + y^2$$

$$\int_0^1 \int_x^1 \sqrt{2 + 4y^2}\, dy\, dx = \int_0^1 \int_0^y \sqrt{2 + 4y^2}\, dx\, dy$$

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$$= \int_0^1 y \sqrt{2 + 4y^2}\, dy = \int_0^4 \frac{1}{8} (2+u)^{\frac{1}{2}} du$$

$$u = 4y^2 \qquad = \frac{1}{8} \cdot \frac{2}{3} (2+u)^{\frac{3}{2}} \Big]_0^4$$

$$du = 8y\, dy \qquad = \frac{1}{12}\left[(6)^{3/2} - 2^{3/2}\right]$$