George M. Bergman
1 Pimentel

Fall 1996, Math 53
Final Exam

11 Dec., 1996
8:00-11:00 AM

1. (40 points) Compute each of the following if it is defined. If an expression is **undefined**, say so. (You do not have to give a reason in such cases.)

(a) (5 points) The unit vector having the same direction as the vector $<-1, 4, 8>$.

(b) (5 points) A parametric equation for the line tangent to the curve $(\sin t, \cos t, e^t)$ at the point where $t = 0$.

(c) (5 points) $D_{\mathbf{u}} f(1,1)$, where $f(x, y) = x/y$, and $\mathbf{u}$ is the vector $<3/5, 4/5>$

(d) (5 points) $\int_x^y \int_y^x 2\, dx\, dy$.

(e) (5 points) An equation of the form $r = f(\theta)$ describing the curve $y = x^2$ in polar coordinates.

(f) (5 points) $\mathrm{grad}\,(x^2 \sin yz)$

(g) (5 points) $\mathrm{curl}\,(\mathrm{div} <x, z, 0>)$

(h) (5 points) $\mathrm{div}\,(\mathrm{curl} <x^2, yz, \sin z>)$

2. (10 points) Let E be a simple solid region in 3-dimensional space, and S its surface, given with the positive orientation. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = <x-y, y-z, x-z>$. Show that the volume of E is equal to $\iint_S \mathbf{F} \cdot d\mathbf{S}$. (Hint: Apply the Divergence Theorem.)

3. (18 points) Let f be a differentiable function of two variables defined in a region D of the plane, and let S be the surface with parametric equations $x = sf(s, t)$, $y = tf(s, t)$, $z = f(s, t)$, where (s, t) ranges over D. Obtain a formula for the area of S in terms of the function f.

4. (12 points) Let R be the triangle in the (u, v) plane defined by the inequalities $u \geq 1$, $u-2 \leq v \leq 2-u$. Let T be the image of R in the (x, y) plane under the transformation given by $x = u^2 - v^2$, $y = uv$. Express the double integral $\iint_T e^{x^2-y^2}\, dA$ as an iterated integral, in the variables u and v. (Do not try to evaluate the integral you get!) You may take it as given that the transformation described above is one-to-one on R.

5. (20 points) Let $\mathbf{F}$ be the vector field $3xz\,\mathbf{i} - 5yz\,\mathbf{j} + z^2\mathbf{k}$.

(a) (7 points) Find constants p and q such that $\mathrm{curl}\,(pyz^2\mathbf{i} + qxyz\,\mathbf{k}) = \mathbf{F}$.

(b) (13 points) Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the part of the surface $z = e^{x^2+y^2}$ below the plane $z = e$, with upward orientation. (Suggestion: Use part (a).)