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Spring 1996, Math 185, Section 1
Second Midterm

19 March, 1996
12:40-2:00 PM

1. (30 points) Let $\Sigma\, a_n z^n$ be a power series with coefficients in $\mathbf{C}$, and let R be a positive real number. Show that the following conditions are equivalent:

(a) $\Sigma\, a_n z^n$ has radius of convergence $\leq R$.

(b) For every positive real number $r < R$ there exists a positive real number c such that $|a_n| \leq c\, r^{-n}$ for $n = 0, 1, 2, \ldots$.

2. (25 points) Let G be a connected open subset of $\mathbf{C}$, and f, g nonzero holomorphic functions on G such that the zeroes of f and the zeroes of g occur at the same points of G, and at each such point, the order of the zero of f is the same as the order of the zero of g. (By a result in the reading, these zeroes are all of *finite* order.) Show that there exists a holomorphic function h on G such that $f = gh$.

(If you can't get the proof in general, you can get 12 points for proving this under the assumption that there is just one point of G where f and g have zeroes, or 18 points for proving it under the assumption that there are exactly two such points. Full credit only for the general case, with possibly infinitely many zeroes.)

3. (a) (25 points) Evaluate $\int_C \frac{e^{iz}}{z^2+1}\, dz$, where C is the circle of radius 1 about the point i, oriented counterclockwise. (One suggestion: Start by writing $1/(z^2+1)$ in the form $a/(z-b) + c/(z-d)$. But there are other ways as well.)

(b) (5 points) For what set of (counterclockwise oriented) circles C is the calculation you made in (a) in fact valid? (No justification required if your answer is correct.)

4. (20 points) Show that there does not exist a holomorphic function f on any neighborhood G of 0 such that for all real $x \in G \cap (0, \infty)$, $f(x) = x^2 \sin 1/x$.