Thursday, March 17, 2005

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ME 106
Mid-term Exam II

Remember, there is no partial credit for answers with incorrect dimensions.

Consider the syringe in Fig. 1 filled with constant density ρ fluid and surrounded by air at constant pressure P_{atm}. From class we know that there are relationships among the force $|\mathbf{F}|$ applied to the left of the plunger (in the x direction), the velocity of the jet of water exiting the syringe V_{exit} with cross-sectional area a, the velocity of the plunger V_{p}, and the cross-sectional area A_{p} of the plunger and of the syringe at the plunger. We used the approximation that the radius of the syringe r was small, so $rg \ll V_{\text{p}}^2/2$, where g is the acceleration of gravity, so that gravitational effects could be ignored while using Bernoulli's equation. We also assumed that the flow was steady in time and that the force applied on the left side of the plunger was equal to the integral of the fluid pressure P_{p} on the right side of the plunger integrated over the area of the plunger.

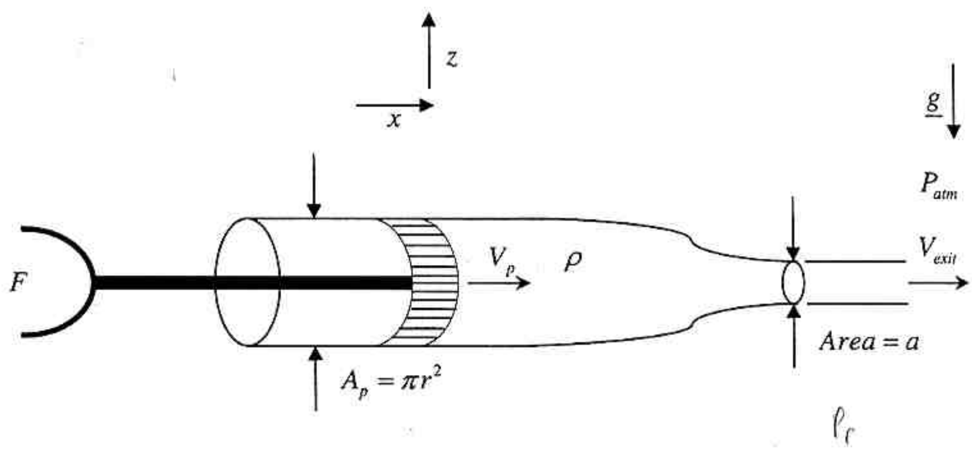

Fig. 1

1) Now consider the same syringe, with the same approximations, but with a leak in the syringe that results in a secondary, exiting jet with cross-sectional area a_{leak}, velocity V_{leak} and at an angle θ_{leak} with respect to the horizontal. See Fig. 2. In terms of the force $|\mathbf{F}|$ applied to the left side of the plunger, what are V_{exit} and V_{leak}, and V_{p}? Hint: Because we are ignoring gravity, P_{p} is constant and $P_{\text{p}} = |\mathbf{F}|/A_{\text{p}}$.

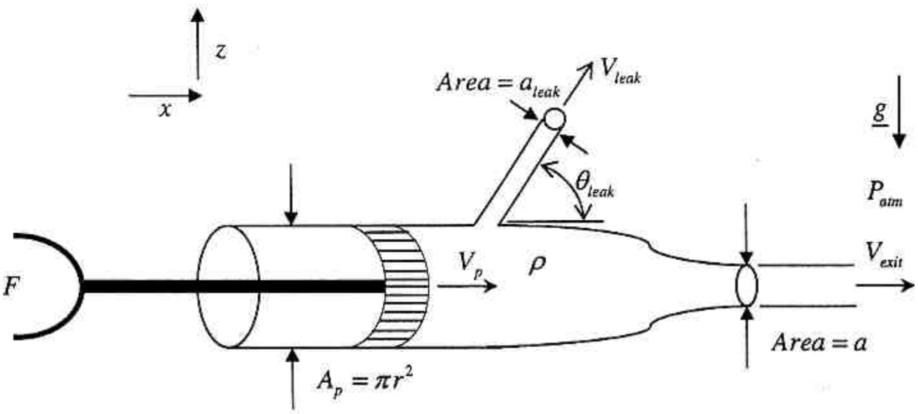

Fig. 2

2) Think carefully before answering this question. If you do it with brute force, you will likely run out of time. Suppose there are now N leaks in the syringe, not just one as in the previous question. Let each leak be a jet of water coming out the side of the syringe with cross-sectional area, a_j, at angle θ_j, and with a velocity V_j, where $j = 1, 2, \cdots, N$. What are V_{exit} and V_{p}? Hint: define $a_{\text{leak}} \equiv \sum_{j=1}^{N} a_j$. Express your answer in terms of a_{leak}.

3) Now consider the super-syringe in Fig. 3, which has *two* plungers. The cross-sectional area of the main syringe is still A_p, and that of the auxiliary with the second plunger is A'_p. The velocity of the main plunger into the syringe is V_p, while the velocity of plunger into the auxiliary tube is V'_p. Find V_exit and also P_p and P'_p, the pressures in the fluid just to the right of the two plungers. Assume that all streamlines exit the syringe with the same velocity V_exit. The average height H of the second plunger above the main part of the syringe is much greater than the radii of the syringe. DO NOT ASSUME THAT H IS SMALL; ITS EFFECT SHOULD BE INCLUDED IN THE BERNOULLI EQUATION.

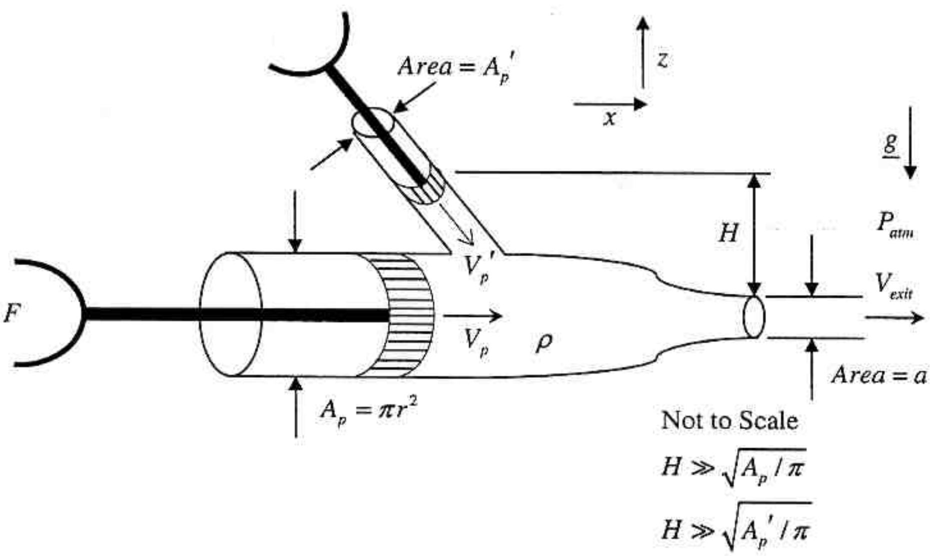

Fig. 3

4) Now consider the simple syringe in Fig. 4. Here the cross-sections of the syringe, the exit jet of water, and the plunger are *square* rather than circular. Let the cross-section of the plunger be $H \times H$ and of the exit jet be $h \times h$. Let H and h be large, so we cannot ignore gravity in the Bernoulli equation. The force applied to the left side of the plunger is $\mathbf{F}$, so we can assume that $\int P_{\mathrm{p}}(z)dydz = |\mathbf{F}|$, where $P_{\mathrm{p}}(z)$ is the pressure of the fluid in the syringe next to the plunger and is a function of z and where the integral is over the cross-section of the syringe. Note that χ is the distance between the bottom of the syringe and the bottom of the exit jet. Because the velocity of the fluid at the plunger is equal to V_{p}, it is constant (independent of z and y). Therefore we can assume that the vorticity of the flow at the plunger as zero, which reduces Euler's equation to:

$$\nabla(V^2/2 + P/\rho + gz) = 0 \rightarrow \partial(V^2/2 + P/\rho + gz)/\partial z = 0 \quad \text{at the plunger.} \qquad (1)$$

Find $P_{\mathrm{p}}(z)$ at the plunger. Hint: P_{atm} is NOT part of the answer, but $|\mathbf{F}|$ is. Find the value of the Bernoulli constant $(V^2/2 + P/\rho + gz)$ along *each* streamline. Find an algebraic equation for V_{p} in terms of $|\mathbf{F}|$, χ and other variables. Hints: V_{exit} is a function of z. Let $z = 0$ at the bottom of the syringe, so the plunger spans $0 \leq z \leq H$ and the exit jet spans $\chi \leq z \leq \chi + h$. For $\alpha \neq 1$, $\int(a + bz)^{\alpha}dz = [(a + bz)^{(1+\alpha)}]/[b(1 + \alpha)]$, where a, b, and α are constants.

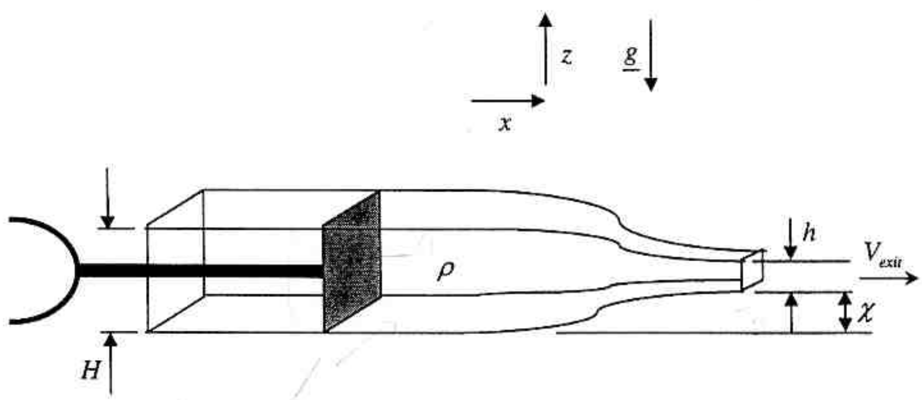

Fig. 4

1 Problem 1

First, mass conservation tells us

$$\rho a V_{exit} + \rho a_{leak} V_{leak} = \rho A_p V_p \tag{1}$$

Constant ρ means

$$a V_{exit} + a_{leak} V_{leak} = A_p V_p \tag{2}$$

Second, using Bernoulli twice (at almost constant height) gives

$$\frac{1}{2}V_{leak}^2 + \frac{P_{atm}}{\rho} \quad = \quad \frac{1}{2}V_p^2 + \frac{P_p}{\rho} \quad = \quad \frac{1}{2}V_{exit}^2 + \frac{P_{atm}}{\rho}$$

That implies

$$\boxed{V_{leak} \quad = V_{exit}}$$

Substitution into Eq.(2) gives

$$\boxed{V_{exit} \quad = \frac{A_p V_p}{a + a_{leak}}}$$

Solving the Bernoulli equation for V_p shows

$$V_p^2 = V_{exit}^2 + 2\frac{P_{atm} - P_p}{\rho} \tag{3}$$

Substituting V_{exit} into Eq.(3)

$$V_p^2 = \left(\frac{A_p V_p}{a + a_{leak}}\right)^2 + 2\frac{P_{atm} - P_p}{\rho} \tag{4}$$

Rearranging

$$V_p^2 \left\{1 - \left(\frac{A_p}{a + a_{leak}}\right)^2\right\} = 2\frac{P_{atm} - P_p}{\rho} \tag{5}$$

$$\boxed{V_p \quad = \quad \sqrt{\frac{2\left(P_{atm} - P_p\right)}{\rho\left\{1 - \left(\frac{A_p}{a + a_{leak}}\right)^2\right\}}} \quad , P_p = |\mathbf{F}|/A_p}$$

Substituting V_p into the boxed equation for V_{exit} gives all the velocities in terms of the given force and areas.

2 Problem 2

For each leak we can consider a streamline emanating from the plunger and exiting the leak. We can write a Bernoulli equation for each case

$$\frac{1}{2}V_{exit}^2 + \frac{P_{atm}}{\rho} \quad = \quad \frac{1}{2}V_j^2 + \frac{P_{atm}}{\rho} \quad = \quad \frac{1}{2}V_p^2 + \frac{P_p}{\rho}, \qquad j = 1, 2, ..., N$$

Similarly to problem (1)

$$\boxed{V_j \quad = V_{exit} \quad , \quad j = 1, 2, ..., N}$$

Mass conservation implies that

$$\rho a V_{exit} + \sum_{j=1}^{N} \rho a_j V_j = \rho A_p V_p \tag{6}$$

Therefore

$$V_{exit}(a + \sum_{j=1}^{N} a_j) = A_p V_p \tag{7}$$

$$V_{exit} = \frac{A_p V_p}{(a + \sum_{j=1}^{N} a_j)} \tag{8}$$

With $a_{leak} \equiv \sum_{j=1}^{N} a_j$, we have exactly the same result for V_{exit} as problem (1)

$$\boxed{V_{exit} \quad = \frac{A_p V_p}{a + a_{leak}}}$$

V_p is found in exactly the same way as question (1), and since we have found V_{exit} to be identical to question (1), V_p is also identical.

3 Problem 3

For this problem, we begin by considering a mass balance over the entire syringe. Because the exit velocity and the two plunger velocities are each constant, the flow through each cross-sectional area is just equal to velocity $\times$ area $\times$ density:

$$\rho v_{exit} a = \rho V_p A_p + \rho V_p' A_p'$$

Since ρ, V_p, V_p', a, A_p, and A_p' are all given, this one equation only contains one unknown:V_{exit}. Solving for V_{exit}, we see that

$$\boxed{V_{exit} = \frac{A_p V_p + A_p' V_p'}{a}}$$

Next, we solve for the next two variables by applying Bernoulli's equation along two different streamlines. First, let's look at a streamline that starts in the main syringe right next to the first plunger and passes through the nozzle to exit to atmospheric pressure. Since we're told to ignore the effects of gravity in this section of the syringe, the resulting relationship is:

$$\frac{V_{exit}^2}{2} + \frac{p_{atm}}{\rho} = \frac{V_p^2}{2} + \frac{p_p}{\rho}$$

From above, we know an expression for V_{exit}, which we substitute in to the equation and solve for p_p:

$$\boxed{p_p = p_{atm} + \frac{\rho}{2}\left(\frac{A_p V_p + A_p' V_p'}{a}\right)^2 - \frac{\rho v_p^2}{2}}$$

Next, let's look at a streamline that starts in the auxiliary tube next to the second plunger (at a height of H) and that passes through the nozzle to exit to atmospheric pressure. Between these two points, the Bernoulli equation gives us that

$$\frac{V_{exit}^2}{2} + \frac{p_{atm}}{\rho} = \frac{V_p'^2}{2} + \frac{p_p'}{\rho} + gH$$

Again, we substitute for V_{exit} and solve for p_p':

$$\boxed{p_p' = p_{atm} + \frac{\rho}{2}\left(\frac{A_p V_p + A_p' V_p'}{a}\right)^2 - \frac{\rho v_p'^2}{2} - \rho g H}$$

4 Problem 4

We are given in the problem statement that the vorticity ω is zero along the plunger. Meanwhile, from Euler's equation we know that

$$\mathbf{v} \times \omega = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{v^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho} + gz\right)$$

We take this problem to be steady flow, so the $\frac{\partial \mathbf{v}}{\partial t}$ term drops out. Now, let's apply this equation to a thin cross-sectional area of the syringe just next to the plunger. In this thin area, the vorticity must be zero for the reasons given in the problem statement. Similarly, in this area, we know that the fluid velocity must be that of the plunger, so $v = v_p$. Then we have that:

$$\begin{aligned}\hat{z} \cdot (\mathbf{v} \times \mathbf{0}) &= \hat{z} \cdot (\nabla(\frac{v_p^2}{2} + \frac{p_p}{\rho} + gz)) \\ 0 &= \frac{\partial}{\partial z}(\frac{v_p^2}{2} + \frac{p_p}{\rho} + gz)\end{aligned}$$

Let's look at what this means. We've just shown that the Bernoulli function has the same value all along the plunger, which means that every streamline starting at the plunger has the same Bernoulli constant. Along each of these streamlines, the value of the Bernoulli function is the same, so the Bernoulli function is constant throughout the flow.

Now, to find p_p, we use the distributive property to break this last relationship up further:

$$0 = \frac{\partial}{\partial z}(\frac{v_p^2}{2}) + \frac{\partial}{\partial z}(\frac{p_p}{\rho} + gz)$$

However, the plunger is a rigid body that's moving axially along the syringe. Its velocity v_p must be constant in z, which means that $\frac{\partial}{\partial z}(\frac{v_p^2}{2}) = 0$. Substituting this in to the above, we see that

$$\frac{\partial}{\partial z}(\frac{p_p}{\rho} + gz) = 0$$

If we integrate this expression in z, we see that

$$p_p(z) = p_0 - \rho g z$$

where p_0 is a constant of integration. To find p_0, we need to make use of the fact that we know the force F that is being applied to the plunger. Because the pressure of the fluid just to the right of the plunger is not constant, we need to integrate over the cross-sectional area of the syringe to obtain a relationship between p_p and F. Letting z = 0 at the bottom of the syringe, we see that:

$$\begin{aligned}|F| &= \int p_p(z) dA \\ &= H \int_0^H p_p(z) dz \\ &= p_0 H^2 - \frac{\rho g H^3}{2}\end{aligned}$$

We solve for p_0:

$$p_0 = \frac{|F|}{H^2} + \frac{\rho g H}{2}$$

Substituting in for p_0, we find that:

$$\boxed{p_p(z) \quad = \tfrac{|F|}{H^2} + \tfrac{\rho g H}{2} - \rho g z}$$

Now that we've solved for p_p, we have enough information to evaluate the Bernoulli function at the plunger and thus, as shown above, find its value throughout the flow in terms of v_p, which is still unknown.

$$\begin{aligned}
\frac{v_p^2}{2} + \frac{p_p}{\rho} + gz &= \frac{v_p^2}{\rho} + \frac{1}{\rho}(\frac{|F|}{H^2} + \frac{\rho g H}{2} - \rho g z) + gz \\
&= \frac{v_p^2}{\rho} + \frac{|F|}{\rho H^2} + \frac{gH}{2}
\end{aligned}$$

We note that this value is, as we expected, a constant. Now that we have the value of Bernoulli's function everywhere in the flow, we use it to find the velocity at the exit.

$$\begin{aligned}
\frac{v_p^2}{\rho} + \frac{|F|}{\rho H^2} + \frac{gH}{2} &= \frac{v_{exit}^2(z)}{2} + \frac{p_{atm}}{\rho} + gz \\
v_{exit}(z) &= \sqrt{v_p^2 + \frac{2|F|}{\rho H^2} + gH - \frac{2p_{atm}}{\rho} - 2gz}
\end{aligned}$$

Finally, to solve for v_p, we need one final equation: that of mass conservation. We remember that v_{exit} is not constant in z, so we must integrate to find the flow rate leaving the syringe through the exit.

$$\begin{aligned}
\rho H^2 v_p &= \rho \int v_{exit} dA \\
&= \rho h \int_\chi^{\chi+h} v_{exit} dz \\
&= \rho h \int_\chi^{\chi+h} \sqrt{v_p^2 + \frac{2|F|}{\rho H^2} + gH - \frac{2p_{atm}}{\rho} - 2gz dz} \\
&= \rho \frac{-h}{3g} [_\chi^{\chi+h} (v_p^2 + \frac{2|F|}{\rho H^2} + gH - \frac{2p_{atm}}{\rho} - 2gz)^{\frac{3}{2}} \\
&= \rho \frac{-h}{3g} [(v_p^2 + \frac{2|F|}{\rho H^2} + gH - \frac{2p_{atm}}{\rho} - 2g(\chi+h))^{\frac{3}{2}} - (v_p^2 + \frac{2|F|}{\rho H^2} + gH - \frac{2p_{atm}}{\rho} - 2g\chi)^{\frac{3}{2}}]
\end{aligned}$$

$$\boxed{\tfrac{3gH^2 v_p}{h} \quad = (v_p^2 + \tfrac{2|F|}{\rho H^2} + gH - \tfrac{2p_{atm}}{\rho} - 2g\chi)^{\frac{3}{2}} - (v_p^2 + \tfrac{2|F|}{\rho H^2} + gH - \tfrac{2p_{atm}}{\rho} - 2g(\chi+h))^{\frac{3}{2}}}$$