MATH 54H FINAL MAY 23, 2002 NAME: L. Bartholdi

Please answer ONLY on the given sheets of paper. No other material will be considered for the grading. Good luck!

JEOPARDY

Please enter, in a maximum of one line, the most appropriate question (with respect to the Math 54H class) corresponding to the answer:

(1) ....

— a sufficient condition is that its eigenvalues are distinct.

(2) ....

— 3.

(3) ....

— hmmm... it's diagonalizable over the complex numbers, but not over the reals.

(4) ....

— it has rank 0.

(5) ....

— there are infinitely many solutions; they are given in parametric form as $x = 4m + 5, y = 5m + 3$.

(6) ....

— it is a differential equation in which there is a special variable, t, and differentiation is done only with respect to t.

(7) ....

— for any $t_0, y_0 \in \mathbb{R}$ there exists a unique differentiable solution $y(t)$ satisfying furthermore $y(t_0) = y_0$.

(8) ....

— the general solution is $y(t) = C \cdot e^{-t}$ for arbitrary $C \in \mathbb{R}$.

Linear Algebra

Consider the following no-chance version of the popular MICRO$OFT game "minesweeper": on a $m \times n$ grid, a certain (unknown) number of mines were placed by your enemy. You don't know where the mines are, but for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ you know the total number $b_{i,j}$ of mines at position (i, j) and at the surrounding 8 squares (or 5 surrounding squares at the boundary of the rectangle, or 3 in the corners).

For which values of m, n is it always possible to deduce from the $b_{i,j}$ the position of the mines? (It may be helpful NOT to assume that there are necessarily 0 or 1 mines per square.)

You may use the following hints. Additional hints may be purchased, at 5% per hint, from your GSI:

- Try first to solve the problem on a $1 \times n$ board; from now on, assume $m = 1$.
- Consider $b = (b_{1,j})$ as a vector, and write $x = (x_j)$ for the original mine positions. Write $b = Ax$ for an appropriate matrix A.
- Show that the problem is solvable if and only if A is invertible.
- Compute $\det(A)$ as a function of n, by induction.
- Show that the $m \times n$ puzzle is solvable if and only if both the $1 \times n$ and the $m \times 1$ puzzle are solvable.

Differential Equations

Show that the earth moves on an ellipse, one of whose focal points is the sun.

You may use the following hints. Additional hints may be purchased, at 5% per hint, from your GSI:

- Gravitation is directed towards the sun, assumed fixed, and decays as the square of the distance to the sun. Hence, if the sun is at $0 \in \mathbb{C}$, and the earth is at $w(t) \in \mathbb{C}$ at time t, the equation is

$$\frac{d^2}{dt^2} w = -G \frac{w}{|w|^3}, \tag{1}$$

  where G is a constant.
- Perform the change of variables $w = z^2$, and show that if z satisfies the equations

$$\frac{d^2}{d\theta^2} z = -z, \quad |z|^2 + \left| \frac{dz}{d\theta} \right|^2 = \frac{G}{2},$$

  where the new time variable θ is given by

$$\theta(t) = \int_0^t \frac{du}{|z(u)|^2},$$

  then w satisfies (1).
- Solve the equation for z, and deduce that z travels on an ellipse centered at 0.
- Show that if z travels on an ellipse centered at 0, then z^2 travels on an ellipse in $\mathbb{C}$ with 0 at its focal point.

HINT LA/1

Each $b_{1,j}$ is a sum of at most three $x_{1,j}$. Write the $b_{1,j}$ as a sum of $x_{1,j}$'s in matrix form.

HINT LA/2

You are asked to find the mine positions $x_{1,j}$. How can you express them in terms of the $b_{1,j}$'s?

HINT LA/3

Write A_n for the matrix in the $1 \times n$ problem. Consider the last column of the matrix A_n. To compute the determinant of A_n, you must compute the determinants of all matrices obtained from striking out the last column, and a row with non-zero entry in last column, in A_n. You should obtain $\det A_n = \det A_{n-1} - \det A_{n-2}$.

HINT LA/3

Consider now the more general problem with m arbitrary. The $(b_{i,j})$ array can again be written as Ax for an unknown array $(x_{i,j})$ of mines. Show that the matrix A (which has dimension $mn \times mn$) is a product of two matrices BC, where B "looks like" A_m and C "looks like" A_n.

HINT DE/1

Express, in complex coordinates, the vector starting from $w(t)$ and pointing towards 0, with length the inverse square of the distance to 0. This is the acceleration to which the earth is subjected.

HINT DE/2

First, using the definition of θ, express $\frac{d}{dt}$ in terms of z and $\frac{d}{d\theta}$. Replace (twice) in $\frac{d^2}{dt^2}w$, and simplify.

HINT DE/3

$z(\theta)$ is defined by a well-understood equation, for which a simple analytic solution exists. Express it using either imaginary exponentials, or sines and cosines.

HINT DE/4

A nice way of writing the ellipse centered at 0, with half-diameters $r + 1/r$ and $r - 1/r$, is as $E_r = \{z + 1/z : |z| = r\}$. This ellipse has focal points at ± 2.

Now the squares of the points in E_r lie on a translate E_{r^2}, which is an ellipse having 0 as a focal point.