Math H53, Honors Multivariable Calculus (Kedlaya, fall 2002)
First midterm exam, Friday, September 27, 2002

The only permitted aid is one $8\frac{1}{2} \times 11$ sheet of paper (one side only). No other notes, calculator, or other assistance are permitted.

There are six problems, each on a separate page, plus an extra page if you need room for scratch work. However, please show all your work on the problem pages; you may continue on the back if you need more space. Work on the scratch page will not be graded.

The total number of points is 150.

Problem 1. This problem concerns the parametric curve given by the equations $x = \cos(t), y = \sin(t)^3$ from $t = 0$ to $t = 2\pi$.

(a) Find the equation for the tangent line to the curve at the point $(x(t_0), y(t_0))$, for those points t_0 where dx/dt and dy/dt are not both zero. (10 points)

(b) Set up a definite integral that computes the area enclosed by the curve. (10 points)

(c) Evaluate that integral. (5 points)

Problem 2. This problem concerns the curve given in polar coordinates by $r = \theta\cos(\theta)$ from $\theta = -\pi$ to $\theta = \pi$.

(a) Find at least ten points on the curve (of your choice); for each point, give its polar and Cartesian coordinates. (Hint: take advantage of symmetry.) (10 points)

(b) Make a rough sketch of the curve. (5 points)

(c) Set up a definite integral that computes the area of the region enclosed by the portion of the curve from $\theta = -\pi/2$ to $\theta = \pi/2$. (10 points)

(d) Evaluate that integral. (5 points)

(c) Set up a definite integral that computes the arc length of the curve from $\theta = -\pi$ to $\theta = \pi$. (Do not attempt to evaluate this integral.) (5 points)

Problem 3. This problem concerns the curve given in polar coordinates by

$$r = \frac{1}{2 - \cos\theta - \sin\theta}.$$

(a) Show that this curve is an ellipse by rewriting the equation given in the form $r = \frac{ed}{1 \pm e\cos(\theta - \alpha)}$. (Hint: write $\cos\theta + \sin\theta$ as $c\cos(\theta - \alpha)$ using the addition formula for cosine.) (10 points)

(b) One focus of this ellipse is at the origin. Specify, in terms of rectangular coordinates, the directrix corresponding to that focus, and the other focus. (5 points)

Problem 4. Let $\mathbf{a}$ be the vector $\langle 3, 1, -1 \rangle$, let $\mathbf{b}$ be the vector $\langle -2, 2, 3 \rangle$, and let $\mathbf{c}$ be the vector $\langle -1, 1, 1 \rangle$.

(a) Compute $\mathbf{a} \cdot \mathbf{b}$. (5 points)

(b) Compute the angle between $\mathbf{a}$ and $\mathbf{b}$, in terms of an inverse trigonometric function. (5 points)

(c) Compute $\mathbf{a} \times \mathbf{b}$. (5 points)

(d) Compute the vector projection of $\mathbf{b}$ onto $\mathbf{a}$. (5 points)

(e) Compute the triple scalar product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and give a geometric interpretation of the result. (10 points)

Problem 5. Let L be the line $x = 1+t, y = 2-t, z = -1+3t$ and let P be the plane $2x-y-z+1 = 0$.

(a) Let Q be the point $(-1,-1,0)$ (which is on P). Find an equation for the plane containing L and Q. (10 points)

(b) Show that line L is parallel to plane P. (5 points)

(c) Compute the distance from L to P. (10 points)

Problem 6.

(a) Convert the point $(r, \theta, \phi) = (2, \pi/3, \pi/4)$ in spherical coordinates into cylindrical and rectangular coordinates. (10 points)

(b) Convert the point $(x, y, z) = (\sqrt{3}/2, -3/2, 1)$ in rectangular coordinates into cylindrical and spherical coordinates. (10 points)