Math 128a Midterm Exam
Oct 10, 2002 K.Hare

NAME (printed) : ______________________
(Family Name) (First Name)

Signature : ______________________

Student Number : ______________________

(1) Do NOT open this test booklet until told to do so

(2) Do ALL your work in this test booklet

(3) SHOW ALL YOUR WORK

(4) CHECK THAT THERE ARE 6 PROBLEMS

(5) NO CALCULATORS

(6) No pushing, biting, or hitting

1	2	3	4	5	6	TOTAL

1 Consider the function

$$f(x) = 2\cos(x) - e^x$$

a: (4 pts) Prove that this function has at least one root between 0 and $\frac{\pi}{2}$.

Notice that $f(0) = 1 > 0$, $f(\pi/2) = -e^{\pi/2} < 0$, and f is continuous. Hence by the Intermediate Value Theorem, there exists a c between 0 and $\pi/2$ such that $f(c) = 0$, which is the desired root.

b: (3 pts) The root of $f(x)$ is actually between 0 and 1. Using the Bisection method, how many steps would it take to determine this root between 0 and 1 to an accuracy of 10^{-3}?

We want $\frac{1-0}{2^n} \le 10^{-3}$ which is equivalent to $2^n \ge 1000$, or $n \ge 10$. So we would need 10 steps of the Bisection method.

c: (3 pts) The calculation of

$$\delta - \sqrt{\delta^2 - 1}$$

is unstable for large δ due to round-off error. Suggest how to rewrite this equation to get a more accurate answer. (Justify your answer.)

Consider

$$\begin{aligned}\delta - \sqrt{\delta^2 - 1} &= (\delta - \sqrt{\delta^2 - 1})\frac{\delta + \sqrt{\delta^2 - 1}}{\delta + \sqrt{\delta^2 - 1}} \\ &= \frac{\delta^2 - \delta^2 + 1}{\delta + \sqrt{\delta^2 - 1}} \\ &= \frac{1}{\delta + \sqrt{\delta^2 - 1}}\end{aligned}$$

This new equivalent formula is more stable, as you are not deleting two nearly equal numbers.

2 a: (3 pts) Define what it means for a sequences $\{p_n\}_{n=0}^{\infty}$ to converge quadratically to p.

We say that p_n converges quadratically to p if

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lambda$$

for some $\lambda \neq 0$.

b: (3 pts) Under what conditions does Newton's method converge quadratically?

Newton's method converges quadratically for a function f if

- $f'(p) \neq 0$
- *f is continuous, and has continuous first and second derivatives.*
- *We start sufficiently close to the root.*

c: (4 pts) Let $p_n = \frac{1}{10^{2^n}}$. What order of convergence does p_n have? (Justify your answer.)

This converges quadratically. First note, $p_n \to 0$.

$$\begin{aligned}\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} &= \lim_{n\to\infty} \frac{|10^{2^{n+1}}|}{|10^{2^n}|^2} \\ &= \lim_{n\to\infty} \frac{10^{2^{n+1}}}{(10^{2^n})^2} \\ &= \lim_{n\to\infty} \frac{10^{2^{n+1}}}{10^{2^{n+1}}} \\ &= \lim_{n\to\infty} 1 \\ &= 1\end{aligned}$$

3: Consider the function

$$f(x) = \frac{x+1}{2}$$

a: (4 pts) Show that $f(x)$ has a unique fixed point p. Find p. Show that the fixed point method converges to p, for all starting points p_0.

Notice that $f(1) = \frac{1+1}{2} = 1$, *so* $p = 1$ *is a fixed point. Consider any interval* $[a, b]$ *where* $a < 1 < b$. *We see that* $f(x) \in [a, b]$ *for all* $x \in [a, b]$, *because* $\frac{a+1}{2} > a$ *and* $\frac{b+1}{2} < b$. *Further notice that* $f'(x) = 1/2$ *for all* $x \in [a, b]$. *Hence the interval* $[a, b]$ *has exactly one fixed point, and the fixed point method will converge to this fixed point for all* $p_0 \in [a, b]$. *Because* a *and* b *are arbitrary, we have that* $f(x)$ *has exactly one fixed point in the real numbers, and that the fixed point method converges for all starting points* p_0.

b: (3 pts) Compute p_1, p_2 and general p_n of the fixed point iteration, given that $p_0 = 0$.

$$\begin{aligned} p_0 &= 0 \\ p_1 &= \frac{1}{2} \\ p_2 &= \frac{3}{4} \\ p_n &= 1 - \frac{1}{2^n} \end{aligned}$$

c: (3 pts) Compute $\hat{p}_0$.

$$\begin{aligned} \hat{p}_0 &= p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0} \\ &= 0 - \frac{(1/2 - 0)^2}{3/4 - 2(1/2) + 0} \\ &= -\frac{1/4}{-1/4} \\ &= 1 \end{aligned}$$

4 a: (3 pts) Assume that a computer system can solve a random 1000×1000 linear system in 3 seconds. How long would you expect the computer system to take to solve a 3000×3000 linear system?

We know that a $n \times n$ *linear system will take* $\mathcal{O}(n^3)$ *time to solve. Thus, if we increase* n *from* 1000 *to* 3000*, we are tripling the size of* n*. Hence the time expected would be* $3^3 \times 3$ *seconds, or 81 seconds.*

b: (2 pts) Assume that a computer system can solve a random 1000×1000 *tridiagonal* system in 3 seconds. How long would you expect the computer system to take to solve a 3000×3000 *tridiagonal* system?

We know that a $n \times n$ *tridiagonal system will take* $\mathcal{O}(n)$ *time to solve. Thus, if we increase* n *from* 1000 *to* 3000*, we are tripling the size of* n*. Hence the time expected would be* 3×3 *seconds, or 9 seconds.*

c: (4 pts) Consider

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 7 & -7 \\ -2 & -7 & -3 \end{bmatrix}$$

Give a LDL^T factorization of A. (Please note, in an LDL^T factorization, the diagonal entries of the L must be 1)

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 7 & -7 \\ -2 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

c: (1 pt) Is A positive definite. Why or why not?

No it is not positive definite. Firstly, the diagonal of a positive definite matrix is positive, A contains a -3. Secondly, the entries of D in a LDL^T *must also be positive, were as here D contains a* -1. *Lastly, the determinate of the leading principal matrices must all be positive, where as the determinate of* $\begin{bmatrix} 2 & 4 \\ 4 & 7 \end{bmatrix}$ *is* -2.

5 a: (5 pts) Consider

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Use Gaussian elimination, with partial pivoting to compute the determinate of A.

We notice that after pivoting, we get

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Performing Gaussian elimination on this gives

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} =: \hat{A}$$

So the determinate of $\hat{A}$ is 4. As there was one row interchange, the determinate of A is -4.

b: (5 pts) Consider the function $f(x)$. Use the information below about $f(x)$, and the initial guesses $x_0 = 1, x_1 = 2$ to compute x_3 and $f(x_3)$ using the Secant method.

x	$f(x)$
1	-1
1.1	$\frac{-89}{100}$
1.2	$\frac{-19}{25}$
1.3	$\frac{-61}{100}$
1.4	$\frac{-11}{25}$
1.5	$\frac{-1}{4}$
1.6	$\frac{-1}{25}$
1.7	$\frac{19}{100}$
1.8	$\frac{11}{25}$
1.9	$\frac{71}{100}$
2.0	1

$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\
&= 2 - \frac{1(2-1)}{1-(-1)} \\
&= 2 - \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \\
&= \frac{3}{2} - \frac{-1/4(3/2-2)}{-1/4-1} \\
&= \frac{3}{2} - \frac{1/8}{-5/4} \\
&= \frac{3}{2} + \frac{1}{10} \\
&= \frac{8}{5}
\end{aligned}$$

$$f(x_3) = \frac{-1}{25}$$

6 a: (2 pts) Define what a diagonally dominate matrix is.

A strictly diagonally dominant matrix A is such that

$$|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$$

for all rows i.

b: (3 pts) Prove or find a counter example. The matrix A is a diagonally dominate matrix if and only if A^T is.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 10 & 100 \end{bmatrix}$$

Clearly A is strictly diagonally dominate, and A^T is not.

c: (2 pts) Define what a permutation matrix is.

A permutation matrix A is an $n \times n$ matrix with exactly one 1 in each row and one 1 in each column. All other entries are 0

d: (3 pts) Prove or find a counter example. The matrix A is a permutation matrix if and only if A^T is.

If A is permutation matrix then A has exactly one 1 in each row, and hence A^T has exactly one 1 in each column. If A is permutation matrix then A has exactly one 1 in each column, and hence A^T has exactly one 1 in each row. If A is a permutation matrix, then all other entries are 0, and hence in A^T, all other entries are 0. Hence if A is a permutation matrix, then A^T is a permutation matrix.

Further if A^T is a permutation matrix, then $(A^T)^T) = A$ is a permutation matrix.