Your name:

Section 3
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Math128a: Numerical Analysis
Final Exam

Write solutions on these sheets.

Put your name on any extra sheet, and turn it in with your exam.

JUSTIFY ALL YOUR ANSWERS.

You can use any result given in class, as soon as you state its contents correctly.

No calculators are allowed.

Good luck!

Computations:

1*. (5 pts)

For what range of values of θ will the approximation $sin\theta \approx \theta$ give results correct to three (**rounded**) decimal places?

2*. (5 pts)

What is the exact value of $x^* - x$ if $x = \sum_{n=1}^{26} 2^{-n}$ and x^* is the nearest **machine number** in the Marc-32?

3*. **(5 pts)**

Use divided differences (show the table) to write the Newton interpolating polynomial for these data:

x	4	2	0	3
$f(x)$	63	11	7	28

4*. **(5 pts)**

State the Contractive Mapping Theorem and show how you would use it to compute a fixed point of the function:

$$F(x) = 4 + \frac{1}{3}\sin 2x.$$

5*. (5 pts)

Discuss the calculation of e^{-x} for $x > 0$ from the series:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$$

Suggest a better way, assuming that the system function for e^x is not available (justify your answer).

6. (5 pts)

Verify that the function $x(t) = \frac{t^2}{4}$ solves the initial-value problem:

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}$$

with one step of the Taylor-series method of order 1 and explain why the numerical solution differs from the solution $x(t) = \frac{t^2}{4}$.

7. (10 pts) Chose **one** of the following problems:

a.Apply the Romberg algorithm to find $R(2,2)$ for the integral:

$$\int_0^{\frac{\pi}{2}} (\frac{x}{\pi})^2 dx$$

(in terms of π).

Hint:$R(n,m) = R(n,m-1) + \frac{1}{4^m-1}[R(n,m-1) - R(n-1,m-1)]$

b.Suppose that $L = lim f(h)$ and that $L - f(h) = c_6 h^6 + c_9 h^9 + \ldots$ What combination of $f(h)$ and $f(\frac{h}{2})$ should be the best estimate of L? How is this process called?

Theory:

8*. **(5 pts)**

How many bits of precision are lost in the substraction $1 - \cos t$ when $t = \frac{1}{4}$?

Hint: $\cos(0.25) = 0.968912$, $2^{-3} = 0.125$, $2^{-4} = 0.0625$, $2^{-5} = 0.03125$, $2^{-6} = 0.015625$, $2^{-7} = 0.007812$, $2^{-8} = 0.003906$. Remember to state any theorem that you use.

9. (15 pts)

a)*List advantages and disadvantages of Newton's form and of Lagrange's form of the interpolating polynomial (in terms of accuracy, the number of operations, the problems for which each form is more suitable etc...)

b)*Define a **trigonometric** polynomial (write the general form). What type of functions can be efficiently interpolated with trigonometric polynomials? What do you mean when you say that a trigonometric polynomial interpolates a function?

c)Give two examples of **single-step methods**. Show the general form of a **linear multistep method**. State the conditions under which the method is implicit, or explicit, give one example for each case.

Problems:

10*. **(10 pts)**

Determine **all** the values of a, b, c, d, e for which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & if \quad x \in (-\infty, 1] \\ c(x-2)^2, & if \quad x \in [1, 3] \\ d(x-2)^2 + e(x-3)^3, & if \quad x \in [3, \infty) \end{cases}$$

next determine the value of the parameters such that the cubic spline interpolates the following table:

x	0	1	4
y	26	7	25

11. (15 points) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 3 & 1 & 1 \end{pmatrix}$

a) Show that A is non-singular (compute the determinant of A).

b) Use the Gaussian elimination with row pivoting to compute the inverse of this matrix. (Show all your steps, display the resulting sequence of elementary matrices).

c) Compute the condition number $\kappa(A)$ of this matrix in the l_∞-norm.

12. (15 points) Prove **one** of the following statements: (10pt)

A^*. Perform two iterations of Newton's method on the system:

$$\begin{cases} xy^2 + x^2y + x^4 = 3 \\ x^3y^5 - 2x^5y - x^2 = -2 \end{cases}$$

starting with $(1, 1)$.

B^*. The Chebyshev polynomials of the first kind are defined recursively as follows:

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)(n \geq 1) \end{cases}$$

Prove of disprove: if n is a divisor of m, then each zero of T_n is a zero of T_m.

C. Using the true solution $x = A\cos t + B\sin t$, find the function ϕ of the shooting method explicitly in this case:

$$\begin{cases} x'' = -x \\ x(0) = 1 \\ x(\frac{\pi}{2}) = 3 \end{cases}$$

If $z_1 = 1$ and $z_2 = 0$, apply the secant method to compute z_3.

Extra credit problem (10pt)

Prove that the Runge-Kutta formula:

$$x(t+h) = x(t) = \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where $F_1 = hf(t,x)$, $F_2 = hf(t+\frac{1}{2}h, x+\frac{1}{2}F_1)$, $F_3 = hf(t+\frac{1}{2}h, x+\frac{1}{2}F_2)$, $F_4 = hf(t+h, x+F_3)$ is of order 4 in the special case that $f(t,x)$ is independent of x. Show that in this case the Runge-Kutta formula is equivalent to Simpson's rule.