Math 128a, Section 3 — Final Exam — December 17, 2001 Prof. Strain

Problem 1 Let

$$f(x,y) = \begin{bmatrix} x^2/4 + y^2/9 - 1 \\ x - y - 1 \end{bmatrix}.$$

(a) Define quadratic convergence of a sequence of vectors x_n to a limit x. **(b)** Compute the Jacobian matrix $Df(x,y)$. **(c)** Determine where $Df(x,y)$ is invertible and compute $Df(x,y)^{-1}$ when it exists. **(d)** Write down Newton's method for solving $f(x,y) = 0$. **(e)** Start with $x_0 = (2,0)^T$ and compute the first two approximations x_1 and x_2 generated by Newton's method. **(f)** Explain why your results demonstrate quadratic convergence.

Problem 2 Consider the iteration

$$x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}.$$

(a) What is it intended to compute? **(b)** Given $a = 2$ and $x_0 = 1$, compute x_1 and x_2. **(c)** Define and determine the order of convergence of this iteration.

Problem 3 Find the QR factorization of

$$A = \begin{bmatrix} 5 & 0 \\ 3 & 5 \\ 4 & 10 \end{bmatrix} = Q_1Q_2Q_3R = \frac{1}{5}P_1\frac{1}{\sqrt{2}}P_2\frac{1}{\sqrt{129}}P_3R.$$

You don't need to multiply together the matrices P_i.

Problem 4 **(a)** Derive a numerical integration formula

$$\int_0^1 f(x)dx = w_0f(0) + w_1f(1) + w_2f(2)$$

which is exact for polynomials of as high degree d as possible, and determine the maximal degree d. **(b)** Without any additional work, determine an equally accurate rule of the form

$$\int_0^1 f(x)dx = u_0f(-1) + u_1f(0) + u_2f(1).$$

(c) Show that

$$\int_0^h f(x)dx = h(w_0f(0) + w_1f(h) + w_2f(2h)) + O(h^{d+1})$$

as $h \to 0$. **(d)** Use (a), (b), and (c) to build a quadrature formula with error $O(h^d)$ on an arbitrary interval $[a,b]$ divided into $n > 1$ subintervals of length $h = (b-a)/n$.

Problem 5 (a) Write down the Newton and Lagrange forms of the quadratic interpolant $p(x)$ to a function f at three points a, b and c. **(b)** Give a formula for the error $p(x) - f(x)$ if f is a nice function with all derivatives bounded. Explain why your error formula makes sense in terms of dimensions, zeroes and the derivatives which appear versus the degree of polynomial used. **(c)** Specialize to $f(x) = R - 1/x$ and evaluate the coefficients in the Newton representation of $p(x)$. **(d)** Use (c) to express p in the power form $p(x) = q_0 + q_1 x + q_2 x^2$. **(e)** How would you use the formula of (d) to derive an iterative method for finding $1/R$?

Problem 6 Suppose A is a square invertible matrix. **(a)** Define the condition number $\kappa(A)$. **(b)** Suppose E is a matrix the same size as A and

$$\kappa(A)\frac{||E||}{||A||} \leq \epsilon \leq \frac{1}{2}.$$

Show that $A + E$ is invertible. **(c)** Show that

$$\frac{||(A+E)^{-1} - A^{-1}||}{||A^{-1}||} \leq 2\epsilon.$$

Extra Credit Problem 7 Given an approximate solution y to the linear system $Ax = b$ with a square invertible matrix A, let $r = b - Ay$ be the residual of y. **(a)** Show that if y satisfies a perturbed linear system $(A + E)y = b$ then the perturbation E must satisfy

$$\frac{||E||}{||A||} \geq \frac{||r||}{||A||||y||}.$$

(b) Show that there is a matrix E such that $(A + E)y = b$ with the norm of E satisfying

$$\frac{||E||}{||A||} = \frac{||r||}{||A||||y||}.$$

(Hint: Try a rank-one matrix $E = \alpha r y^T$ for some well-chosen scalar α.) **(c)** Define the backward relative error in an approximate solution y of $Ax = b$. **(d)** Show that an approximate solution y of $Ax = b$ has backward relative error $O(\epsilon)$ if and only if it has a residual r satisfying

$$\frac{||r||}{||A||||y||} = O(\epsilon).$$

Extra Credit Problem 8 Prove that any model of floating-point arithmetic which requires that the floating-point result of the multiplication $x*y$ be given by the exact result correctly rounded satisfies the relative error bound

$$\frac{|x * y - \text{fl}(x * y)|}{|x * y|} \leq \epsilon$$

as long as no overflow or underflow occurs and $x * y \neq 0$.