Math 114 Final 2003 May 23. R. Borcherds

Please make sure that your name is on everything you hand in.
You are allowed calculators and 1 sheet of notes.
All questions have about the same number of marks.

1. Find polynomials $a(x), b(x)$ in $Q[x]$ such that

$$(x^4 + x)a(x) + (x^2 + 1)b(x) = 1$$

2. Prove that there exist irreducible polynomials over Q of arbitrarily large degree. (One way to do this is to use Eisenstein's criterion.)

3. If α is a root of $x^3 - 3x + 1$ show that $1/(1 - \alpha)$ is also a root. Use this to show that over any field $x^3 - 3x + 1$ is either irreducible or splits into the product of 3 linear factors. What is the Galois group of this polynomial over Q?

4. Find the Galois group of the splitting field of $x^4 - 2$ over Q. What is its order? Is it abelian? Is it cyclic?

5. Give an example of a field of characteristic $p > 0$ such that the Frobenius automorphism $x \mapsto x^p$ is not onto.

6. Prove that the Galois group of $GF(p^n) : GF(p)$ is cyclic of order n, and describe a generator of it. ($GF(p^n)$ is the finite field of order p^n.)

7. Find the conjugacy classes of the dihedral group D_{10} of order 10.

8. What is the Galois group of $x^7 - 1$ over Q?

9. Construct a field with 16 elements.

10. Prove that the additive group of any finite field of order p^n (p prime) is a product of n cyclic groups of order p.