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Spring 2001, Math 114
Final Examination

15 May, 2001
12:30-3:30 PM

1. (20 points, 4 points each.) Complete each of the following definitions. (Do not give examples or other additional facts about the concepts defined.)

(a) If K is a field and $f(t)$, $g(t)$ are nonzero elements of $K[t]$, then a *highest common factor* of $f(t)$ and $g(t)$ means

(b) If $L:K$ is a field extension, then the *degree* of L over K (written $[L:K]$) means

(c) If $L:K$ is a field extension and H is a subgroup of $\Gamma(L:K)$, then $H^\dagger$ is defined as

(d) If L is a finitely generated extension of the field K, then the *transcendence degree* of L over K (also written tr.deg.$(L:K)$) means

(In answering this, you may assume facts and/or definitions preceding the definition of transcendence degree in either Stewart or my handout on the subject. However, do not refer as Stewart does to "the number defined in" a certain lemma of his, without re-stating the conditions of that lemma.)

(e) If K is a field of characteristic $p \neq 0$, then the *Frobenius endomorphism* of K (in Stewart's language, the 'Frobenius monomorphism') is

2. (20 points; 4 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated.)

(a) A field extension $L:K$ and an intermediate field M (i.e., a field with $K \subseteq M \subseteq L$) such that $[L:K] = 3$ and $[M:K] = 2$.

(b) A field extension $L:K$ such that the characteristic of K and the characteristic of L are not equal.

(c) A finite inseparable field extension.

(d) An irreducible quintic polynomial (i.e., polynomial of degree 5) $f(t) \in \mathbb{Q}[t]$ which is solvable in radicals.

(e) A polynomial $f(t) \in \mathbb{Q}[t]$ which has noncommutative Galois group over $\mathbb{Q}$, but commutative Galois group over $\mathbb{Q}(\omega)$, where ω is a primitive cube root of unity.

3. (7 points) Suppose $L:K$ is an algebraic extension, τ an element of $\Gamma(L:K)$, and x an element of L. Show that x and $\tau(x)$ have the same minimal polynomial over K.

4. (15 points) Let p be a prime number and n a positive integer. Show that the irreducible factors of the polynomial $t^{p^n} - t \in \mathbb{Z}_p[t]$ are precisely all irreducible polynomials $f \in \mathbb{Z}_p[t]$ whose degrees are divisors of n.

(Hint: Consider the extension of $\mathbb{Z}_p$ generated by a zero of an irreducible polynomial f. You may use the fact, proved in homework, that the subfields of $\mathbf{GF}(p^n)$ are precisely the fields $\mathbf{GF}(p^m)$ for which m divides n.)

5. (20 points) Recall that a field L is said to be *algebraically closed* if every polynomial over L of degree > 0 splits in L.

(a) (8 points) Suppose L is an algebraically closed field, and K any subfield of L. Show that if E is a finite algebraic extension of K, then E can be mapped into L by a K-monomorphism. (Hint: If E is normal over K, you can use uniqueness of splitting fields.)

(b) (8 points) Show that if E is a proper algebraic extension of the field $\mathbb{R}$ of real numbers (i.e., if E is algebraic over $\mathbb{R}$ and not equal to $\mathbb{R}$), then E is isomorphic over $\mathbb{R}$ (i.e., isomorphic by an $\mathbb{R}$-isomorphism) to $\mathbb{C}$.

(Suggestion: first do this in the case where E is finite over $\mathbb{R}$, with the help of part (a), then deduce that the case where E is infinite over $\mathbb{R}$ cannot occur.)

(c) (4 points) Give an example of a proper finitely generated extension $E:\mathbb{R}$ which is not isomorphic over $\mathbb{R}$ to $\mathbb{C}$, and give a reason why it is not.

6. (18 points) (a) (9 points) Suppose L is a field of characteristic not equal to 2, in which the polynomial $t^4 + 1$ splits. Show that the subgroup of the multiplicative group of L generated by the zeros of that polynomial is cyclic of order 8. If we let α be a generator of that group, which elements α^i $(0 \leq i < 8)$ are the zeros of $t^4 + 1$?

(b) (9 points) Suppose K is a field of characteristic not equal to 2, in which the polynomial $t^4 + 1$ is irreducible. Determine the Galois group of $t^4 + 1$ over K precisely, writing out its multiplication table. (In writing this table, specify the meanings of the symbols you use for the group elements. Your discussion should show why the Galois group must have this table, though you don't have to show all arithmetic calculations. Hint: If α is one zero of the given polynomial, part (a) shows what elements a member of the Galois group can take α to.)