* −5 AT LEAST FOR ANY MATHEMATIC ERRORS IN WORK

* NO CREDIT FOR DIMENSIONALLY INCORRECT WORK

UNIVERSITY OF CALIFORNIA, BERKELEY
MECHANICAL ENGINEERING
ME106 Fluid Mechanics
1st Test, S08 *Prof S. Morris*

NAME SOLUTIONS

Exam stats
Mean: 124.8/200
St. Dev: 37.1

1.(65) **(a)** Find the fluid acceleration **a** for plane stagnation point flow in which the velocity is given by the expression $\mathbf{V} = cx\mathbf{i} - cy\mathbf{j}$ (the constant $c > 0$).

EITHER

$$\frac{d\underset{\sim}{V}}{dt} = \frac{d}{dt}(cx\underset{\sim}{i}) - \frac{d}{dt}(cy\underset{\sim}{j})$$

$$= c\underset{\sim}{i}\frac{dx}{dt} - c\underset{\sim}{j}\frac{dy}{dt}$$

($c\underset{\sim}{i}$, $c\underset{\sim}{j}$ const in mag. & direction)

But $\frac{dx}{dt} = cx$, $\frac{dy}{dt} = -cy$ $\Rightarrow$ $\boxed{\frac{d\underset{\sim}{V}}{dt} = c^2(\underset{\sim}{i}x + y\underset{\sim}{j})}$

OR. $\underset{\sim}{V} = u\underset{\sim}{i} + v\underset{\sim}{j}$ $\qquad \frac{d\underset{\sim}{V}}{dt} = u\frac{\partial \underset{\sim}{V}}{\partial x} + v\frac{\partial \underset{\sim}{V}}{\partial y}$ $\qquad \because \frac{\partial \underset{\sim}{V}}{\partial t} = 0$

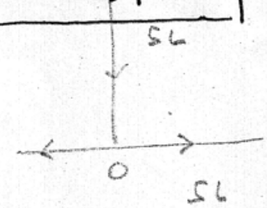
$\frac{\partial \underset{\sim}{V}}{\partial x} = c\underset{\sim}{i}$, $\frac{\partial \underset{\sim}{V}}{\partial y} = -c\underset{\sim}{j}$ $\Rightarrow$ $\frac{d\underset{\sim}{V}}{dt} = (cx)(c\underset{\sim}{i}) + (-cy)(-c\underset{\sim}{j})$

$\Rightarrow \boxed{\underset{\sim}{a} = c^2(\underset{\sim}{i}x + y\underset{\sim}{j})}$

Lost points for:
math errors, −5
did not underline unit vectors, −5
confuse vectors and scalars, −10

Pr. 1 Stats
Mean: 57.4/65
St Dev: 12.1

(b) By considering the stagnation streamline, explain why $\mathbf{a} \neq 0$ in this flow.



On the branch $x=0$ $\quad \underset{\sim}{V} = -cy\underset{\sim}{j}$
Velocity of FP moving along SL to origin decreases with y $\Rightarrow$ $\underset{\sim}{a} \neq 0$.

No credit for arguments that:
- velocity changes throughout flow
- using $\underline{a}$ w/o physical reasoning

IN BLOCK LETTERS PRINT YOUR NAME ON THIS PAGE

Pr. 2 Stats
Mean: 40.6/65
St. Dev: 22.1

2. (65) The vector field $\mathbf{V} = -\frac{1}{2}cr\hat{\mathbf{r}} + v(r,t)\hat{\theta} + cz\hat{\mathbf{k}}$ ($c > 0$, constant) represents a vortex that is being stretched in the axial direction. For this flow, the fluid acceleration is given by the expression

$$\mathbf{a} = \left(\tfrac{1}{4}c^2 r - \frac{v^2}{r}\right)\hat{\mathbf{r}} + \left(\frac{\partial v}{\partial t} - \tfrac{1}{2}c\frac{\partial(rv)}{\partial r}\right)\hat{\theta} + c^2 z\hat{\mathbf{k}}.$$

(You are not required to show that.) Find the partial differential equation satisfied by the function $v(r,t)$. (You are not asked to solve it.)

Given. In cylindrical polar coordinates r, θ, z, if $\mathbf{F} = f_r\hat{\mathbf{r}} + f_\theta\hat{\theta} + f_z\hat{\mathbf{k}}$ is an arbitrary vector, then

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f_r & rf_\theta & f_z \end{vmatrix}.$$

Euler's equation for constant ρ is satisfied $\Leftrightarrow$ $\boxed{\nabla \times \underset{\sim}{a} = 0}$ +20

$$r\nabla\times\underset{\sim}{a} = \begin{vmatrix} \hat{r}, & r\hat{\theta}, & \hat{k} \\ \frac{\partial}{\partial r}, & 0, & 0 \\ \frac{c^2 r}{4} - \frac{v^2}{r}, & r\left\{\frac{\partial v}{\partial t} - \frac{1}{2}c\frac{\partial}{\partial r}(rv)\right\}, & c^2 z \end{vmatrix} = 0\hat{r} - r\hat{\theta}\frac{\partial}{\partial r}r\left(\frac{\partial v}{\partial t} - \frac{1}{2}c\frac{\partial}{\partial r}(rv)\right) + 0\hat{k}$$

+30 Expanding $\nabla\times\underset{\sim}{a} = 0$

so $\nabla\times\underset{\sim}{a} = 0$ $\Longleftrightarrow$ $\frac{\partial}{\partial r}r\left(\frac{\partial v}{\partial t} - \frac{1}{2}c\frac{\partial}{\partial r}(rv)\right) = 0$ A.

i.e. $\boxed{\frac{\partial}{\partial t}(rv) - \frac{1}{2}cr\frac{\partial}{\partial r}(rv) = f(t)}$ B.

Either A. or B. is acceptable. + 15 for complete and correct solution.

Point deductions:
- Did not see $\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial z} = 0$
-15

* If you attempted to use Euler's equation and plugged in CORRECTLY, +30

- Missing r terms,
-5

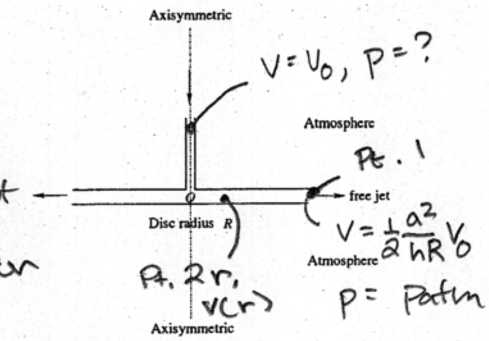
Pr. 3 Stats
Mean: 26.8/70
St Dev: 17.8

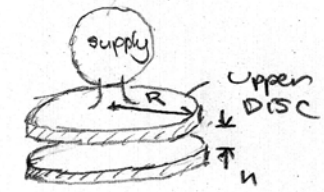
3. ($\cancel{85}$ 70) The figure shows the side view of two discs each of radius R, and separated by fixed distance h. Air enters the gap between the discs with speed V_0 through the supply tube of radius a, flows radially and leaves as a free jet into the atmosphere. Atmospheric pressure is p_a. Within the gap, the speed V varies with distance r from the axis according to the rule

$$V = \begin{cases} \frac{1}{2}\frac{r}{h}V_0 & \text{if } r < a \\ \frac{1}{2}\frac{a^2}{hr}V_0 & \text{if } r > a. \end{cases}$$

(You are not required to show that.) The pressure within the supply tube is unknown; it is *not* atmospheric.

* If you used Bernoulli w/ one pt @ supply WITH P = Patm, you got a max of +5 for (a)





(40 max) **(a)** Using the Bernoulli equation, find the pressure p within the gap as a function of r, ρ, V_0, h and a. Sketch p as a function of r showing clearly any maxima, minima and zeros. (You need to consider the cases $r < a$, $r > a$ separately.)

BE along SL from r to exit

Pt 2 Pt 1

$$\Rightarrow \quad \frac{1}{2}\rho V^2(r) + p(r) = \frac{1}{2}\rho V_e^2 + p_a$$

$$\Rightarrow \quad p(r) - p_a = \frac{1}{2}\rho V_e^2 \left\{ 1 - \frac{V^2(r)}{V_e^2} \right\}$$

+20 For properly using Bernoulli properly. This includes those who tried to find Psupply

But $\boxed{V_e = \frac{1}{2}\frac{a^2}{hR}V_0}$ $\Rightarrow$ $\frac{V}{V_e} = \begin{cases} \frac{Rr}{a^2}, & r<a \\ \frac{R}{r}, & r>a \end{cases}$

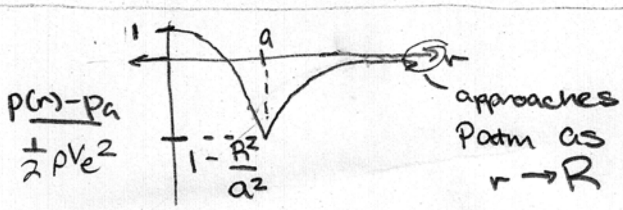
$$\Rightarrow \quad \boxed{\frac{p(r)-p_a}{\frac{1}{2}\rho V_e^2} = \begin{cases} 1 - \frac{R^2}{a^4}r^2 \quad +5, & r<a \\ 1 - \frac{R^2}{r^2} \quad +5, & r>a \end{cases}}$$

BOXED EQNS GIVE THE SOL'N

1s08–3

since $R > a$

$\frac{R}{a} > 1$



+5 for correct plot

+5 for correct labeling + comments

$\left(\frac{30}{70}\right)$ **(b)** Noting that the flow is axisymmetric, derive the integral expressing the force F exerted by the air on both sides of the lower disc in terms the pressure difference $p(r) - p_a$.

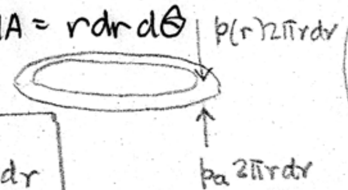
Area element is a ring of radius r, thickness dr.

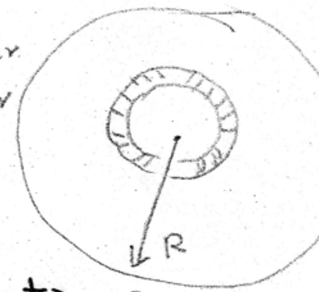
REQUIRED FOR FULL CREDIT $\boxed{dA = 2\pi r\, dr}$ or $dA = r\,dr\,d\theta$

Vertical force on dA $= 2\pi\{p_a - p(r)\}r\,dr$

$$\boxed{\text{Force upward on lower disc} = 2\pi\int_0^R \{p_a - p(r)\}r\,dr}$$

-10 for Failing to consider p_a underneath





(c) Hence show that F is given by the expression

$$F = \tfrac{1}{4}\pi\rho\frac{a^4}{h^2}V_0^2\left(\ln\frac{R}{a} - \tfrac{1}{4}\right).$$

Explain physically why, despite the presence of a stagnation point at O, the force is upwards if $R \gg a$.

$$\frac{F}{\rho\pi V_e^2} = -\int_0^R \left(1 - \frac{V^2}{V_e^2}\right) r\,dr$$

$$= -\int_0^a \left(1 - \frac{R^2}{a^4}r^2\right) r\,dr - \int_a^R \left(1 - \frac{R^2}{r^2}\right) r\,dr$$

$$= -\left[\cancel{\tfrac{1}{2}r^2} - \frac{R^2}{4a^4}r^4\right]_0^a - \left[\cancel{\tfrac{1}{2}r^2} - R^2\ln r\right]_a^R$$

$$= R^2\ln\frac{R}{a} - \frac{1}{4}R^2$$

Subst for V_e

i.e. $$\frac{F}{\frac{1}{4}\rho\pi\frac{a^4}{h^2R^2}V_0^2} = R^2\left(\ln\frac{R}{a} - \frac{1}{4}\right)$$

$$\Rightarrow \boxed{\frac{F}{\frac{1}{4}\pi\rho\frac{a^4}{h^2}V_0^2} = \ln\frac{R}{a} - \frac{1}{4}}$$

$+5$

- Requires full + correct integration of correct V function

END

$+5$

Although near stagnation $p(r) > p_a$, over most of the disc is $p(r) < p_a$. Consider mass conservation. For $r > a$, as r incr, v decr. $\therefore$ P increases, but must reach p_a at jet $\therefore$ $p(r) < p_a$ for $r > a$