MATH 113 FINAL MAY 22, 2002 NAME: L. Bartholdi

First, a Little Joke

There once were a Math 104 and a Math 113 student, who decided after their finals to go to Africa for their summer holidays. They were at a hut in the savannah, when a lion attacked the Math 104 student. He ran frantically around the hut, and the lion also ran after him around the hut. The Math 113 student yelled to his friend "Beware! The lion's almost on you!", to which the Math 104 student replied "Don't worry! I'm three turns ahead!".

Explanation of the joke: the Math 104 student, obsessed with Real Analysis, fails to realize that he should measure his distance to the lion in the group $\mathbb{R}/\mathbb{Z}$, rather than in $\mathbb{R}$.

Now please answer the following questions on these sheets of paper:

Group theory

(1) Consider the group $G = (\mathbb{R}, +)$, i.e. the set $\mathbb{R}$ with addition as operation. Prove that all cyclic subgroups of G are of the form $\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}_+$.

All cyclic subgroups H of G are normal, and G/H is isomorphic to $\mathbb{R}/\mathbb{Z}$. Give an isomorphism between G/H and $\mathbb{R}/\mathbb{Z}$.

Consider the group $F = (\mathbb{R}^*_+, \cdot)$, i.e. the set of positive real numbers with multiplication as the operation. Show that F is isomorphic to G, and give an isomorphism.

F is a subgroup of $E = (\mathbb{R}^*, \cdot)$. Explain why F is normal in E, and describe the quotient E/F.

(2) Consider the symmetric group S_n of n symbols. What is the order of S_n?

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What is the exponent (i.e. the smallest $e \in \{1, 2, 3, \ldots\}$ such that $x^e = 1$ for all x in the group) of S_1, S_2, S_3, S_4, S_5? ______________

The elements of S_n can be described by square matrices in such a way that matrix multiplication corresponds to the group operation. (Such a matrix description is called a *linear representation*). Give a generating set of S_3, and then describe a linear representation of S_3 by giving the matrix corresponding to each generator.

Hint: your computations will be easier if you pick your generating set to be as small as possible, and the matrices to be of small size.

(3) Did you like the joke? ______________ If not, give a better joke:

Ring theory

(4) Let X be a set, and consider the structure $R = (\mathcal{P}(X), +, \cap)$, i.e. the set of subsets of X, with as addition operation the symmetric difference $A + B = (A \cup B) \setminus (A \cap B)$, and as multiplication the intersection of sets. Prove that R is a ring.

Is R commutative? ______________

Does R have a unity? ______________ If so, say what the unity is; otherwise, describe a ring containing R and a unity.

Is R integral? ______________ If no, give a divisor of 0 in R; otherwise, describe the field of quotients of R.

How many units are there in R? ______________

Does R have a characteristic? If so, what is it? ______________

(5) Consider the structure $A = (\mathbb{R}, \min, +)$, i.e. the set $\mathbb{R}$, with addition of $a, b \in \mathbb{R}$ defined as the minimum of a and b, and multiplication defined as the (usual) sum of a and b.

Is A a ring? ______________ If so, say whether A is commutative and/or integral. If not, say which axioms A satisfies and which it fails to satisfy.

(6) Consider the ring $B = \mathbb{Z}/18\mathbb{Z}$. Describe all ideals I of B, and for each of them describe the quotient ring B/I.

Is B commutative? ______________ Does it have a unity? ______________

Is it integral? ______________ If not, give a divisor of 0: ______________

(7) Consider the ring $D = M_2(\mathbb{R})$. Show that D is neither commutative nor integral.

Give a subgroup of the group of units of D that is isomorphic to S_3.

Give a subring of D isomorphic to the ring $\mathbb{C}$ of complex numbers.