Math 113 Midterm #1, 2/27/03, 8:00 – 9:30 AM M. Hutchings

To receive full credit you must **justify all answers** except where otherwise stated. However you do not need to write long, ultra-detailed proofs. A few sentences should suffice for each question. The point is to demonstrate that you understand the material. (If you finish early, you can always go back and add more details just to be sure.)

No books, notes, calculators, collaboration, or other aids are permitted. There are 5 pages and a total of 50 points. Please write your answers on the exam, not in a blue book. You may use the backs of the pages if necessary.

Good luck!

1. (5 points each)

   (a) How many injective functions are there from $\{1,2,3\}$ to $\{4,5,6,7,8\}$?

   (b) Find integers x and y with $103x + 113y = 1$. (No justification required, just show your work.)

2. (10 points) Let G be a group. Define a relation $\sim$ on G as follows: $x \sim y$ if and only if there exists $a \in G$ such that $axa^{-1} = y$. (In this case we say that x and y are *conjugate*.) Prove that $\sim$ is an equivalence relation.

3. (5 points each) The "quaternion group" Q has 8 elements $1, -1, i, -i, j, -j, k, -k$ and the following multiplication table:

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

(a) True or false: $Q \simeq \mathbb{Z}_8$.

(b) True or false: $Q \simeq D_4$. Hint: consider elements x with $x^2 = e$.

(For this question, you do not have to write the proof that something is a structural property. Aside from that, justify your answers as usual.)

4. (5 points each) Consider the following permutations x and y in S_8:

$$x = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8), \qquad y = (2\ 3\ 4\ 5).$$

(a) Express xy as a product of disjoint cycles. (No justification required.)

(b) What is the order of x (i.e. the smallest positive integer k such that $x^k = e$)?

5. (5 points each)

   (a) True or false (i.e. prove, or give a counterexample and justify): if H_1 and H_2 are subgroups of a group G, then their *union*

   $$H_1 \cup H_2 = \{x \mid x \in H_1 \text{ or } x \in H_2\}$$

   is also a subgroup of G.

   (b) What is the order of the subgroup of $\mathbb{Z}_{999}$ generated by 498? (Hint: find a smaller generator of this subgroup.)