Math H113 Professor K. A. Ribet
Final Exam May 16, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

1. Let H be a subgroup of a finite group G. For each g in G, consider the subset $S_g := H(gHg^{-1})$ of G; this is the subset HK where $K = gHg^{-1}$. Show that H is normal in G if and only if all the sets S_g have the same size.

2. Let p and q be distinct primes. Show that every group of order p^2q has a normal Sylow subgroup. [You can assume that p^2q is different from 12, since we studied groups of order 12 in class.]

3. Let G be a finite abelian group. Suppose that the intersection of all non-identity subgroups of G is a non-identity subgroup of G. Prove that G is isomorphic to $\mathbf{Z}/p^n\mathbf{Z}$ for some prime p and some positive integer n.

4. Let A be a non-empty set and let G be a subgroup of the group S_A of permutations of A. For $a \in A$, let $G_a = \{ g \in G \mid ga = a \}$. Show that $G_{ga} = gG_ag^{-1}$ for $g \in G$, $a \in A$. If G acts transitively on A, show that $\bigcap_{g \in G} gG_ag^{-1} = \{1\}$ for each $a \in A$.

Further, if G is an abelian subgroup of S_A that acts transitively on A, show that $G_a = \{1\}$ for all $a \in A$. Prove that $|G| = |A|$ in this case.

5. Consider the evaluation homomorphism $\varphi : \mathbf{R}[x] \to \mathbf{C}$ that sends each polynomial $f(x)$ to the complex number $f(2 + 3i)$. Find a generator for the kernel of φ.

6. Let n be a positive integer, and let p be a prime number that divides $2^n + 1$. If m is an odd positive integer, show that p does not divide $2^m - 1$.

7. Is an irreducible element of an integral domain necessarily prime? (Give a proof or a counter-example.)

If R is a commutative ring with 1, it is true that the intersection of two maximal ideals of R is a prime ideal? (Proof or counter-example.)

If R is a commutative ring with 1, is it necessarily true that $1 - x$ is a unit if $x^9 = 0$? (Proof or counterexample.)

8. Let $n \geq 3$ be an odd integer. Show that the dihedral group D_{2n} of order $2n$ has exactly $(n + 3)/2$ conjugacy classes.

Find a finite group G and a subgroup H of G so that H has more conjugacy classes than G.